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Integrability of solutions of the Skorokhod Embedding Problem for diffusions.

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Abstract

Suppose X is a time-homogeneous diffusion on an interval $I^X \subseteq \mathbb{R}$ and let μ be a probability measure on I^X. Then τ is a solution of the Skorokhod embedding problem (SEP) for μ in X if τ is a stopping time and $X_\tau \sim \mu$.

There are well-known conditions which determine whether there exists a solution of the SEP for μ in X. We give necessary and sufficient conditions for there to exist an integrable solution. Further, if there exists a solution of the SEP then there exists a minimal solution. We show that every minimal solution of the SEP has the same first moment.

When X is Brownian motion, there exists an integrable embedding of μ if and only if μ is centred and in L^2. Further, every integrable embedding is minimal. When X is a general time-homogeneous diffusion the situation is more subtle. The case with drift can be reduced to the local martingale case by a change of scale. If Y is a diffusion in natural scale, and if the target law is centred, then as in the Brownian case, there is an integrable embedding if the target law satisfies an integral condition. However, unlike in the Brownian case, there exist integrable embeddings of target laws which are not centred. Further, there exist integrable embeddings which are not minimal. Instead, if there exists an integrable embedding, then the set of minimal embeddings is the set of embeddings such that the mean equals a certain quantity, which we identify.

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1 Introduction

Let X be a regular, time-homogeneous diffusion on an interval $I^X \subseteq \mathbb{R}$, with $X_0 = x \in int(I^X)$, and let μ be a probability measure on $\overline{I^X}$. Then τ is a solution of the Skorokhod embedding problem (Skorokhod [24]) for μ in X if τ is a stopping time and $X_\tau \sim \mu$. We call such a stopping time an embedding (of μ in X).

For a general Markov process Rost [22] gives necessary and sufficient conditions which determine whether a solution to the Skorokhod embedding problem (SEP) exists for a given target law. The conditions are expressed in terms of the potential. When applied to Brownian motion (where we include the case of Brownian motion on an interval subset of $\mathbb{R}$, provided the process is absorbed at finite endpoints) these conditions lead to a characterisation of the set of measures which can be embedded in Brownian

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motion. Then, in the case of a regular, one-dimensional, time-homogeneous diffusion with absorbing endpoints, necessary and sufficient conditions for the existence of a solution to the SEP can be derived via a change of scale. Let s be the scale function of X; then $Y = s(X)$ is a local martingale, and in particular a time-change of Brownian motion. Further, let $I = s(I^X)$ be the state space of Y. Then the set of probability measures for which a solution of the SEP exists depends on both I and the relationship between the starting value of Y and the mean of the image under s of the target law, see Theorem 2.1 below.

Apart from the existence result above, most of the literature on the SEP has concentrated on the case where X is Brownian motion in one dimension. Exceptions include Bertoin and LeJan [4] who consider embeddings in any time-homogeneous Markov process with a well-defined local time, Grandits and Falkner [10] (drifting Brownian motion), Hambly *et al* [12] (Bessel process of dimension 3) and Pedersen and Peskir [18] and Cox and Hobson [8] (these last two consider embeddings in a general, one dimensional, time-homogenenous diffusion).

In the Brownian setting many solutions of the SEP have been described; see Obloj [16] or Hobson [13] for a survey. Given there are many solutions, it is possible to look for criteria which characterise 'small' or 'good' solutions. In both the Brownian case and more generally, there is a natural class of good solutions of the SEP, namely the minimal embeddings (Monroe [15]). An embedding τ is minimal (for μ in X) if whenever $\sigma \leq \tau$ is another embedding (of μ in X) then $\sigma = \tau$ almost surely.

Another criteria for a good solution might be that it is integrable, or as small as possible in the sense of expectation. In this article we are interested in the integrability or otherwise of solutions of the SEP, and the relationship between integrability and minimality in the case where X is a time-homogeneous diffusion in one dimension.

Consider the case where X is Brownian motion null at zero and write W for X. By the results of Rost [22] there exists a solution of the SEP for μ in W for *any* probability measure μ on $\mathbb{R}$. If we require integrability of the embedding then the story is also well-known:

Theorem 1.1 (Shepp [23], Monroe [15])**.** *There exists an integrable solution of the SEP for μ in W if and only if μ is centred and in L^2. Further, in the case of centred square-integrable target measures, τ is minimal for μ if and only if τ is an embedding of μ and $\mathbb{E}[\tau] < \infty$.*

Our goal in this paper is to consider the case where X is a regular time-homogeneous diffusion on an interval I^X with absorbing endpoints. Let $x \in int(I^X)$ denote the initial value of X, and let μ be a probability measure on $\overline{I^X}$.

Our main result is as follows (note that the explicit form for $E_X(x;\mu)$ is given in (5.1) below):

Theorem 1.2. *There exists an integrable solution of the SEP for μ in X if and only if $E_X(x;\mu) < \infty$ where $E_X(x;\mu)$ is a function of the scale function, speed measure and initial value of X and target law μ. Further, in the case where $E_X(x;\mu) < \infty$ then τ is minimal for μ if and only if τ is an embedding of μ and $\mathbb{E}[\tau] = E_X(x;\mu)$.*

In the Brownian case if μ is not centred, or if it is centred but not in L^2, then there is no intregrable embedding. If μ is centred and in L^2 then there is a dichotomy, and for any embedding either $\mathbb{E}[\tau] = \int x^2\mu(dx)$ or $\mathbb{E}[\tau] = \infty$. Hence, if the target law is centred and square integrable then minimality of an embedding is equivalent to integrability. This is not true in general for diffusions: we can have integrable embeddings which are not minimal. The converse is also true: both in the Brownian case and more generally we can have minimal embeddings which are not integrable. This will be the case if $E_X(x;\mu) = \infty$.

For a general, regular, time-homogeneous diffusion, the first step is to switch to natural scale. Hence, much of our analysis considers the case of a diffusion $Y = (Y_t)_{t\geq 0}$ in natural scale. If $Y_0 = y$ and if the target law has mean y, then the first result is that there exists an embedding which is integrable if and only if the target law ν satisfies an integral condition $\int q(x)\nu(dx) < \infty$, generalising the L^2 condition for the Brownian case. Here q is a function with the property that $q(Y_t) - t$ is a local martingale, see (2.3) for a definition. The second result is that in the case where an integrable embedding exists, an embedding τ of ν is minimal if and only if $\mathbb{E}[\tau] = \int q(x)\nu(dx)$.

In the centred case the proofs of Root [21] and Monroe [15] extend with only minor modification, see Sections 3.1 and 3.2. So, the main innovation of this paper is to cover the non-centred case. (Consideration of a process like upward drifting Brownian motion started at zero, and a target law which is a point mass at $z > 0$, shows that the non-centred case is not a pathological situation, but rather the generic case.) In the non-centred case for Brownian motion there are no integrable embeddings and so there are no results which our formulae must nest as special cases. Instead we find that the condition on the target law for the existence of an integrable embedding has two parts: an integral element as in the centred case, and a growth condition at infinity which depends on ν only through its mean. If this condition is satisfied then an embedding is minimal if and only if its expected value is equal to an expression $E_Y(y;\nu)$ which depends on the target law and the speed measure.

This article makes two further contributions. First, we consider the existence or otherwise of integrable embeddings in a diffusion started at an entrance-not-exit point, and conditional on the existence of an integrable embedding, give a necessary and sufficient condition for minimality.

Second, we outline a technique for proving minimality even when integrability fails. This technique is useful in the Brownian case for a target law $\nu \in L^1$ which is not centred. The idea is that minimality of an embedding τ^W of ν in W is preserved under time-change. Hence, after a change of speed measure, we can consider the minimality or otherwise of τ^Y which is modification of τ^W, in a process Y which is a time-change of W. Since we make no change of scale τ^Y is an embedding of ν in Y. Since $\nu \in L^1$, for some choice of process Y we will have $E_Y(0;\nu) < \infty$. We have a criteria for the minimality of τ^Y, and hence τ^W is minimal for ν in W if and only if τ^Y is minimal in Y if and only if $\mathbb{E}[\tau^Y] = E_Y(0;\nu)$. We illustrate this method of proving minimality by considering the extension of the Azéma-Yor [3] stopping time to non-centred target laws due to Cox and Hobson [8].

The remainder of the paper is structured as follows. In the next section we state the main results, focussing on the case of a diffusion in natural scale started at an interior point. Section 3 is devoted to a proof of these results. In Section 4 we extend the analysis to include processes started at a boundary point. Then in Section 5 we show how the conclusions can be adapted to the case of diffusions not in natural scale. Finally in Section 6 we explain our ideas about proving minimality of embeddings even when integrability fails.

We close the introduction by considering a quartet of illuminating and motivating examples.

Example 1.3. *Let $Z = (Z_t)_{t\geq 0}$ be Brownian motion on $\mathbb{R}_+$ absorbed at zero, and with $Z_0 = z > 0$. Then there exists an embedding of μ if and only if $\int x\mu(dx) \leq z$. Moreover, there exists an integrable embedding of μ in Z if and only if $\int x\mu(dx) = z$ and $\int x^2\mu(dx) < \infty$ and then an embedding τ is minimal if and only if $\mathbb{E}[\tau] < \infty$ if and only if*

$\mathbb{E}[\tau] = \int (x-z)^2 \mu(dx)$. Note that Z is a supermartingale so the necessity of $\int x\mu(dx) \le z$ is clear.

Example 1.4. *Let $V = (V_t)_{t \ge 0}$ be upward drifting Brownian motion with $V_0 = v$. In particular, suppose V solves $V_t = v + aW_t + bt$ with $b > 0$ and $W_0 = 0$, and set $\beta = 2b/a^2$. Then there exists an embedding of μ if and only if $\int e^{-\beta(u-v)}\mu(du) \le 1$. (Upward drifting Brownian motion is transient to $+\infty$ and so there will be an embedding of μ provided μ does not place too much mass at values far below v.) Moreover, there exists an integrable embedding of μ if and only if $\int e^{-\beta(u-v)}\mu(du) \le 1$ and $\int u^+\mu(du) < \infty$. If there exists an integrable embedding then an embedding τ is minimal if and only if $\mathbb{E}[\tau] = E(v;\mu)$ where*

$$E(v;\mu) = \frac{1}{b}\left(\int u\mu(du) - v\right) < \infty.$$

Example 1.5. *Let $P = (P_t)_{t\ge 0}$ be a Bessel process of dimension 3 started at $P_0 = p > 0$. Then there exists an embedding of μ if and only if $\int x^{-1}\mu(dx) \le p^{-1}$. Moreover, there exists an integrable embedding of μ if and only if $\int x^{-1}\mu(dx) \le p^{-1}$ and $\int x^2\mu(dx) < \infty$ and then an embedding τ is minimal for μ if and only if τ is an embedding and $\mathbb{E}[\tau] = E(p;\mu)$ where*

$$E(p;\mu) = \frac{1}{3}\int x^2\mu(dx) - \frac{p^2}{3}. \tag{1.1}$$

Note that a Bessel process is transient to infinity, and so for there to exist an embedding of μ, μ cannot place too much mass near zero. For an integrable embedding then in addition we cannot have too much mass far from zero as the process takes a long time to get there. Note also that $Y = P^{-1}$ is a diffusion in natural scale and that Y is the classical Johnson-Helms example of a local martingale which is not a martingale.

The results extend to the case $p = 0$. Then any μ on $\mathbb{R}^+$ can be embedded in P. There exists an integrable embedding if and only if μ is square integrable.

Example 1.6. *Let $Q = (Q_t)_{t\ge 0}$ solve $dQ_t = (1 + Q_t^2)dW_t$ subject to $Q_0 = 0$. Let $\mu = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$. Let $\tau = \max[\inf\{u : Q_u = -1\}, \inf\{u : Q_u = 1\}]$. Then τ is an embedding of μ and τ is integrable, but τ is not minimal.*

2 Preliminaries and main results for diffusions in natural scale

For a diffusion process $Z = (Z_t)_{t\ge 0}$, let $H^Z_z = \inf\{s \ge 0 : Z_s = z\}$, and let $H^Z_{a,b} = H^Z_a \wedge H^Z_b$. Where the process Z involved is clear, the superscript may be dropped.

Let Y be a diffusion in natural scale with state space I and with $Y_0 = y \in int(I)$. Denote the endpoints of I by $\{\ell, r\}$ with $-\infty \le \ell < y < r \le \infty$. Suppose that if Y can reach an endpoint of I, then such an endpoint is absorbing. Suppose that Y is regular, ie for all $y' \in int(I)$ and $y'' \in I$, $\mathbb{P}^{y'}(H^Y_{y''} < \infty) > 0$. The diffusion Y in natural scale is characterised by its speed measure which we denote by m. Recall that if Y solves the SDE $dY_t = \eta(Y_t)dB_t$ for a continuous diffusion coefficient η then $m(dy) = dy/\eta(y)^2$.

Let ν denote a probability measure on $\overline{I}$. Provided $\nu \in L^1$, write $\overline{\nu}$ for the mean of ν, with a similar convention for other measures.

We reserve the labels X and μ for a diffusion which is not in natural scale, and the corresponding target distribution, see Section 5.

Theorem 2.1 (Pedersen and Peskir [18], Cox and Hobson [8]). (i) *Suppose I is a finite interval. Then ν can be embedded in Y if and only if $y = \int x\nu(dx)$.*

(ii) *Suppose $I = (\ell, \infty)$ or $[\ell, \infty)$ for $\ell > -\infty$. Then ν can be embedded in Y if and only if $y \ge \int x\nu(dx)$.*

(iii) *Suppose* $I = (-\infty, r)$ *or* $(-\infty, r]$ *for* $r < \infty$. *Then* ν *can be embedded in* Y *if and only if* $y \leq \int x\nu(dx)$.

(iv) *Suppose* $I = \mathbb{R}$. *Then any* ν *can be embedded in* Y.

The idea behind the proof is to write Y as a time-change of Brownian motion, $Y_t = W_{\Gamma_t}$. Then, since Y is absorbed at the endpoints we must have that $\Gamma_t \leq H^W_{\ell,r}$ for each t.

In the first case of the theorem Y is a bounded martingale and $\mathbb{E}[Y_\tau] = y$ for any τ. In the second case Y is a local martingale bounded below and hence a supermartingale for which $\mathbb{E}[Y_\tau] \leq y$. In the third case Y is a submartingale.

2.1 Discussion of the Brownian case and Theorem 1.1

For W a Brownian motion null at 0, $W^2_{t\wedge\tau} - (t \wedge \tau)$ is a martingale and

$$\mathbb{E}[\tau] = \liminf \mathbb{E}[t \wedge \tau] = \liminf \mathbb{E}[W^2_{t\wedge\tau}] \geq \mathbb{E}[\liminf W^2_{t\wedge\tau}] = \mathbb{E}[W^2_\tau]. \tag{2.1}$$

Moreover, from Doob's L^2 submartingale inequality we know that $\mathbb{E}[\tau] < \infty$ if and only if $\mathbb{E}[(\max_{0\leq s\leq\tau}|W_s|)^2] < \infty$, and then $(W_{t\wedge\tau})_{t\geq 0}$ and $(W^2_{t\wedge\tau})_{t\geq 0}$ are uniformly integrable. It follows that if $\mathbb{E}[\tau] < \infty$ then

$$0 = \lim \mathbb{E}[W_{t\wedge\tau}] = \mathbb{E}[W_\tau] = \int x\mu(dx)$$

and

$$\mathbb{E}[\tau] = \lim \mathbb{E}[t \wedge \tau] = \lim \mathbb{E}[W^2_{t\wedge\tau}] = \mathbb{E}[W^2_\tau] = \int x^2\mu(dx),$$

so that μ is centred and in L^2.

Conversely, if μ is centred and in L^2 then there are several classical constructions which realise an integrable embedding, including those of Skorokhod [24] and Root [21]. See Obloj [16] or Hobson [13] for a discussion.

The final statement of Theorem 1.1 is deeper, and follows from Theorem 5 of Monroe [15]. One of the main goals of this work is to extend the work of Monroe to general diffusions. Note that the arguments above yield that in the Brownian case if τ is an embedding of μ and $\mathbb{E}[\tau] < \infty$ then $\mathbb{E}[\tau] = \int x^2\mu(dx)$, so that if μ is centred and in L^2 then every integrable embedding is minimal.

2.2 Diffusions in natural scale

Consider now the case of a general diffusion Y in natural scale. Suppose $Y_0 = y = 0$ and that ν is centred. Then to determine whether there might exist a integrable embedding we might expect to replace the condition $\int x^2\mu(dx) < \infty$ of the Brownian case with some other integral test depending on the speed measure m of Y and the target measure ν. Indeed we find this is the case with x^2 replaced by a convex function q defined in (2.3) below.

But what if ν is not centred? In the Brownian case there is no hope that the target law can be embedded in integrable time, not least because $\mathbb{E}[H^W_x] = \infty$ for each non-zero x, but what if Y is some other diffusion?

Suppose the state space I of Y is unbounded above. Suppose $Y_0 = y$ and $\nu \in L^1$ with $\overline{\nu} = \int x\nu(dx) < y$. One candidate way to embed ν is to first wait until $H^Y_{\overline{\nu}} = \inf\{t : Y_t = \overline{\nu}\}$ and then to embed ν in Y started at $\overline{\nu}$, ie to set

$$\tau = H^Y_{\overline{\nu}} + \tau^{\overline{\nu},\nu} \circ \Theta_{H^Y_{\overline{\nu}}} \tag{2.2}$$

where Θ is the shift operator $\Theta_t(\omega(\cdot)) = \omega(t + \cdot)$ and $\tau^{\overline{\nu},\nu}$ is some embedding of ν in Y started at $\overline{\nu}$. Note that since I is unbounded above and Y is a time-change of

Brownian motion, it follows that $H^Y_{\overline{\nu}}$ is finite almost surely. The embedding in (2.2) will be integrable if *both* $H^Y_{\overline{\nu}}$ *and* $\tau^{\overline{\nu},\nu}$ are integrable, and we can decide if it is possible to choose $\tau^{\overline{\nu},\nu}$ integrable using the integral test of the centred case. Our results show that although embeddings of ν need not be of the form given in (2.2), nonetheless there exist integrable embeddings if and only if both $\mathbb{E}[H^Y_{\overline{\nu}}] < \infty$ and there is an integrable embedding $\tau^{\overline{\nu},\nu}$ of ν in Y started at $\overline{\nu}$. In that case every minimal embedding has the same first moment.

Definition 2.2. *Let Y be a regular diffusion in natural scale on $I \subseteq \mathbb{R}$. Suppose $Y_0 = y$. Let m denote the speed measure of Y. Define q_u via*

$$q_u(z) = 2\int_u^z dv \int_u^v m(dw) = 2\int_u^z m((u,v))dv \tag{2.3}$$

and let $q = q_y$.

Note that an alternative expression for q_u is $q_u(z) = 2\int_u^z (z-w)m(dw)$. Further, $q(Y_t) - t$ is a local martingale, null at zero.

Definition 2.3. *If $\nu \notin L^1$ set $E_Y(y;\nu) = \infty$. For $\nu \in L^1$ define*

$$E_Y(y;\nu) = \int q_y(z)\nu(dz) + \lim_{n\to\infty} |y - \overline{\nu}| \frac{q_y(y + n\mathrm{sign}(y - \overline{\nu}))}{n} \tag{2.4}$$

In the centred case the limit in (2.4) is necessarily zero, and it does not matter what convention we use for $\mathrm{sign}(0)$. In the non-centred case, if $\overline{\nu} < y$ then if there is to be an embedding of ν we must have that I is not bounded above. Then $\lim_{n\to\infty} \frac{q_y(y+n)}{n} = 2m((y,\infty))$ exists in $(0,\infty]$ by the convexity of q_y. If $\overline{\nu} < y$ and I is bounded above then we cannot take the limit in (2.4), and E is not defined (we could define it to be infinite), but this does not matter since it is not possible to embed ν in Y started from y. From the remarks above we have that $E_Y(y,\nu)$ has an alternative representation

$$E_Y(y;\nu) = \int q_y(z)\nu(dz) + 2(y-\overline{\nu})m((y,\infty))\mathcal{I}_{\{y>\overline{\nu}\}} + 2(\overline{\nu}-y)m((-\infty,y))\mathcal{I}_{\{y<\overline{\nu}\}} \tag{2.5}$$

where $\mathcal{I}_A$ denotes the indicator function of A.

The second term in (2.4) arises as the difference between the two sides in an application of Fatou's Lemma, and is a consequence of the non-uniform integrability of $(Y_{t\wedge\tau})_{t\geq 0}$ in the non-centred case.

In the case of a diffusion in natural scale, the main result of this paper is the following:

Theorem 2.4. *There exists an integrable solution of the SEP for ν in Y if and only if $E_Y(y;\nu) < \infty$. Further, in the case where $E_Y(y;\nu) < \infty$ we have that τ is minimal for ν if and only if τ is an embedding and $\mathbb{E}[\tau] = E_Y(y;\nu)$.*

3 Every minimal embedding has the same first moment

Our goal is to prove Theorem 2.4. We begin with a useful lemma which gives two simple sufficient criteria for minimality. The second one has a stronger hypothesis, but leads to stronger conclusions which cover both minimality and integrability.

Lemma 3.1. *Suppose $-\infty \leq \ell < L < y < R < r \leq \infty$.*

(i) Suppose that at most one endpoint of I is infinite. If $\tau < H^Y_{\ell,r}$ then τ is minimal for $\mathcal{L}(Y_\tau)$.

(ii) Suppose $\tau \le H^Y_{L,R}$. Then τ is minimal for $\mathcal{L}(Y_\tau)$ and $\mathbb{E}[\tau] = \mathbb{E}[q(Y_\tau)] = E_Y(y, \mathcal{L}(Y_\tau))$.

(iii) If ρ is any embedding of ν then $\mathbb{E}[\rho] \ge \int q_y(z)\nu(dz)$. If also ν is centred then $\mathbb{E}[\rho] \ge E_Y(y,\nu)$.

Proof. (i) We prove the result in the case $I = (\ell,\infty)$ or $[\ell,\infty)$ with $\ell > -\infty$. The other cases are similar.

Since I has a finite endpoint, Y is transient. Further, Y is a supermartingale.

Let $\nu \sim \mathcal{L}(Y_\tau)$ so that τ is an embedding of ν. Then $\overline{\nu} \le y$. Let $\sigma \le \tau$ be another embedding. Then, from the supermartingale property, $\mathbb{E}[Y_\tau; Y_\sigma \le x] \le \mathbb{E}[Y_\sigma; Y_\sigma \le x]$ and since Y_σ and Y_τ are equal in law,

$$\mathbb{E}[x - Y_\tau; Y_\sigma \le x] \ge \mathbb{E}[x - Y_\sigma; Y_\sigma \le x] = \mathbb{E}[x - Y_\tau; Y_\tau \le x] = \sup_A \mathbb{E}[x - Y_\tau; A]$$

Then, modulo null sets $(Y_\tau \le x) = (Y_\sigma \le x)$ and hence $Y_\sigma = Y_\tau$ almost surely.

Suppose $\sigma \le \eta \le \tau$. Then

$$Y_\eta \ge \mathbb{E}[Y_\tau|\mathcal{F}_\eta] = \mathbb{E}[Y_\sigma|\mathcal{F}_\eta] = Y_\sigma,$$

almost surely. But also $\mathbb{E}[Y_\eta - Y_\sigma] \le 0$ since Y is a supermartingale, and hence $Y_\eta = Y_\sigma$ almost surely. It follows that Y is almost surely constant over the interval $[\sigma,\tau]$. But Y is a time change of Brownian motion $Y_t = W_{\Gamma_t}$ for some strictly increasing time-change Γ. Brownian motion has no intervals of constancy, and hence nor does Y. It follows that $\sigma = \tau$ almost surely and hence τ is minimal.

(ii) If $\tau \le H_{L,R}$ then we have $Y_{t\wedge\tau}$ is bounded and $\mathbb{E}[Y_\tau] = y$. Also q is bounded on $[L,R]$. Hence $E_Y(y, \mathcal{L}(Y_\tau)) = \mathbb{E}[q(Y_\tau)]$ and

$$\mathbb{E}[q(Y_\tau)] = \lim_t \mathbb{E}[q(Y_{t\wedge\tau})] = \lim_t \mathbb{E}[t\wedge\tau] = \mathbb{E}[\tau].$$

If $\tau \le H_{L,R}$ and $\rho \le \tau$ and both ρ and τ are embeddings of $\mathcal{L}(Y_\tau)$, we must have $\mathbb{E}[\rho] = \mathbb{E}[\tau]$ and hence $\rho = \tau$ almost surely. Hence τ is minimal. See also Proposition 4 in [1].

(iii) Let $(L_n)_{n\ge1}$ and $(R_n)_{n\ge1}$ be monotonic sequences with $L_n < y < R_n$ and $L_n \downarrow \ell$, $R_n \uparrow r$. Then from Fatou's Lemma we have

$$\mathbb{E}[\rho] = \lim_n \mathbb{E}[\rho \wedge H_{L_n,R_n}] = \liminf_n \mathbb{E}[q(Y_{\rho\wedge H_{L_n,R_n}})] \ge \mathbb{E}[q(Y_\rho)] = \int q(x)\nu(dx).$$

□

3.1 The centred bounded case

Suppose ν is a measure with mean $\overline{\nu} = y$ and support in a subset $[L,R] \subset (\ell,r)$ of I where $L < y < R$.

Suppose that σ is an embedding of ν. Our goal is to show that there exists an embedding $\tilde\sigma$ of ν such that $\tilde\sigma \le \sigma \wedge H_{L,R}$. Then $\tilde\sigma$ is minimal and $\mathbb{E}[\tilde\sigma] = \int q(x)\nu(dx)$. It follows that if σ is minimal, then $\sigma = \tilde\sigma$ and $\mathbb{E}[\sigma] = \int q(x)\nu(dx)$.

Following a definition of Root [21], we define a barrier to be a closed subset B of $G = [0,\infty] \times [-\infty,\infty]$ such that:
- $(\infty, x) \in B$ for all $x \in [-\infty,\infty]$,
- $(t,\ell) \cup (t,r) \in B$ for all $t \in [0,\infty]$,
- if $(0,x) \in B$ for $x > y$ then $(0,x') \in B$ for $x' > x$, similarly if $(0,x) \in B$ for $x < y$ then $(0,x') \in B$ for $x' < x$ and finally
- if $(t,x) \in B$ then $(s,x) \in B$ for all $s > t$.

Let $\mathcal{B}$ be the space of all barriers and given L,R with $\ell \le L < y < R \le r$ let $\mathcal{B}_{[L,R]}$ be the set of all barriers B with $(0, L)$ and $(0, R)$ in B. Then $(t, x) \in B$ for $(t \geq 0, x \leq L)$ and $(t \geq 0, x \geq R)$.

Root [21] describes a topology such that $\mathcal{B}$ (and hence also $\mathcal{B}_{[L,R]}$) is compact. For $B \in \mathcal{B}$ define

$$\tau_B = \inf\{t : (t, Y(t)) \in B\}.$$

Lemma 3.2. *Suppose ν has mean y and support in $[L, R]$. Suppose that σ is an embedding of ν. Then there is a barrier $B \in \mathcal{B}_{[L,R]}$ such that $\sigma \wedge \tau_B \leq H_{L,R}$ is a minimal embedding of ν and $\mathbb{E}[\sigma \wedge \tau_B] = \int q_y(x)\nu(dx)$.*

Proof. The proof follows the proof of Lemma 4 in Monroe [15] which in turn is based on the proof of Lemma 2.2 in Root [21].

The idea is first to consider the case where ν has finite support, and to construct a barrier $B \in \mathcal{B}_{[L,R]}$ such that $\sigma \wedge \tau_B \leq H_{L,R}$ is a minimal embedding of ν.

We can then deduce the result for general, centred ν with support in $[L, R]$ by approximation, using a sequence of measures $(\nu_n)_{n\geq 1}$ with finite support, with associated barriers $(B_n)_{n\geq 1}$. From the fact that $\mathcal{B}_{[L,R]}$ is compact we find a barrier B_∞ such that $\sigma \wedge \tau_{B_\infty} \leq H_{L,R}$ is a minimal embedding of ν.

In the modified proof we make use of the fact that Y is continuous and $\mathbb{E}[H^Y_{L,R}] < \infty$. Otherwise the main change relative to Lemma 4 of [15] is that we make use of Lemma 3.1 to argue that for any embedding τ with $\tau \leq H^Y_{L,R}$ we have $\mathbb{E}[\tau] = E_Y(y, \nu)$. □

For a diffusion Y with state space I, speed measure m and initial value $Y_0 = y$, and for a law ν on $[L, R]$ with mean y, we have that $E_Y(y; \nu) = \int q_y(x)\nu(dx)$. Clearly $E_Y(y; \nu) < \infty$ under the present conditions on ν.

Corollary 3.3. *Suppose ν has mean y and support in $[L, R] \subset (\ell, r)$. Then an embedding σ of ν is minimal if and only if $\mathbb{E}[\sigma] = E_Y(y; \nu)$.*

Proof. By the first case of Theorem 2.1 there exists an embedding σ of ν in Y, and then by Lemma 3.2 there exists a minimal embedding $\tilde{\sigma} = \sigma \wedge \tau_B$ with $\mathbb{E}[\tilde{\sigma}] = E_Y(y; \nu)$. If σ is minimal then $\sigma = \tilde{\sigma}$ and $\mathbb{E}[\sigma] = E_Y(y; \nu)$. Conversely, by the arguments at the end of Lemma 3.1, for any embedding $\mathbb{E}[\sigma] \geq E_Y(y; \nu)$ and so if $\mathbb{E}[\sigma] = E_Y(y; \nu)$ then σ is minimal. □

3.2 The general centred case

Now suppose that ν is centred but that there is no subset $[L, R] \subset (\ell, r)$ for which $\nu([L, R]) = 1$. We cannot follow the proof in Monroe [15] exactly, since that proof makes use of the fact that Brownian motion is recurrent. Instead we construct a sequence of measures $(\nu_n)_{n\geq n_0}$ with supports in bounded intervals $[L_n, R_n] \subset (\ell, r)$ and such that $(\nu_n)_{n\geq n_0}$ converges to ν. Hence, given σ and ν_n there is a barrier B_n with associated stopping time $\tilde{\sigma}_n = \tau_{B_n} \wedge \sigma$ such that $Y_{\tilde{\sigma}_n}$ has law ν_n. For our choice of approximating sequence of measures we argue that the sequence of stopping times τ_{B_n} is monotonic increasing with limit τ_∞. Finally we show that $\sigma \wedge \tau_\infty$ is minimal and embeds ν.

The main issue is to show that the barriers have a monotonicity property, and hence that the stopping times τ_{B_n} are monotonic, and have a limit. Again we are guided by the proof of Lemma 4 in Monroe [15].

Recall that our current hypothesis is that ν is a measure on $\overline{I}$ such that $\nu \in L^1$ and $Y_0 = y = \overline{\nu}$.

For a measure $\eta \in L^1$ with mean c and support in $[\ell, r]$ define the potential $U_\eta : [\ell, r] \mapsto \mathbb{R}_+$ via $U_\eta(x) = \mathbb{E}^{Z\sim\eta}[|Z - x|]$. Let $\mathcal{V}_c$ be the set of convex functions $f : [\ell, r] \mapsto \mathbb{R}$ satisfying $f(x) \geq |x - c|$, together with $\lim_{x\downarrow\ell}\{f(x) - (c - x)\} = 0 = \lim_{x\uparrow r}\{f(x) - (x - c)\}$.

Then $U_\eta \in \mathcal{V}_c$ and there is a one-to-one correspondence between elements of $\mathcal{V}_c$ and probability measures on $[\ell, r]$ with mean c, see, for example, Chacon and Walsh [7]. For a pair of probability measures η_i with support in $[\ell, r]$ we have that η_1 is less than or equal to η_2 in convex order, and write $\eta_1 \leq_{cx} \eta_2$ if and only if $U_{\eta_1}(x) \leq U_{\eta_2}(x)$ for all $x \in [\ell, r]$.

Given ν, fix $n_0 \geq 1/U_\nu(\overline{\nu})$. For $n \geq n_0$ define $U_n : [\ell, r] \mapsto \mathbb{R}_+$ via

$$U_n(x) = \max\{U_\nu(x) - 1/n, |x - \overline{\nu}|\},$$

and let ν_n be the probability measure with potential U_n. Then there exist $\{a_n, b_n\}$ such that $[a_n, b_n] \subset (\ell, r)$, $\nu_n(A) = \nu(A)$ for all measurable subsets $A \subset (a_n, b_n)$ and $\nu_n([\ell, a_n)) = 0 = \nu_n((b_n, r])$. Then ν_n has atoms at a_n and b_n and mean $\overline{\nu}$, see, for example Chacon and Walsh [7]. Further $(a_n)_{n\geq n_0}$ and $(b_n)_{n\geq n_0}$ are monotonic sequences and the family $(\nu_n)_{n\geq n_0}$ is increasing in convex order.

Theorem 3.4. *Suppose $\nu \in L^1$ and $Y_0 = y = \overline{\nu}$. Let σ be an embedding of ν. There exists an barrier B such that $\tau_B \wedge \sigma$ also has law ν and $\mathbb{E}[\tau_B \wedge \sigma] = E_Y(y; \nu)$ where $E_Y(y; \nu) = \int q(z)\nu(dz)$.*

Proof. For each n, fix ν_n as above. From our study of the bounded case we know there is a barrier B_n which we can assume contains $\{(t, x), x \leq a_n$ or $x \geq b_n\}$ such that $Y_{\tau_{B_n}\wedge\sigma}$ has law ν_n.

It follows from arguments in Monroe [15, p1296-7] that if $p > n$ then we may assume $B_p \subseteq B_n$. Define $B_\infty = \cap B_n$ and set $\tau_\infty = \tau_{B_\infty}$. Then $\tau_{B_n} \uparrow \tau_\infty$. Also $\tau_{B_n} \wedge \sigma \uparrow \tau_\infty \wedge \sigma$ and

$$\mathcal{L}(Y_{\tau_\infty\wedge\sigma}) = \lim \mathcal{L}(Y_{\tau_{B_n}\wedge\sigma}) = \lim \nu_n = \nu.$$

It only remains to prove that $\mathbb{E}[\sigma \wedge \tau_\infty] = E_Y(y; \nu)$. But from the monotonicity in convex order of ν_n,

$$\mathbb{E}[\sigma \wedge \tau_\infty] = \lim \mathbb{E}[\sigma \wedge \tau_{B_n}] = \lim E_Y(y; \nu_n) = \lim \int q(z)\nu_n(dz) = \int q(z)\nu(dz).$$

□

3.3 The uncentred case

Without loss of generality we may assume that the mean of ν satisfies $\overline{\nu} < y$. Then for there to be an embedding of ν we must have that I is unbounded above.

The idea is to construct a sequence of measures $(\nu_n)_{n\geq n_0}$ with supports in bounded intervals $[L_n, R_n] \subset (\ell, \infty)$ and such that $(\nu_n)_{n\geq n_0}$ converges to ν, and then to use results from the bounded, centred case to deduce results for the general case.

Recall that ν is a measure on $\overline{I}$ such that $\nu \in L^1$. Let F_ν be the distribution function of ν and let F_ν^{-1} denote the right inverse. In particular, if $U \sim U[0, 1]$ then $F_\nu^{-1}(U)$ has law ν.

Lemma 3.5. *Suppose $y > \overline{\nu}$. There exists a family of probability measures $(\nu_n)_{n\geq n_0}$ with the properties that*

1. *for each $n \geq n_0$, ν_n has support in $[a_n, b_n] \subseteq (\ell, \infty)$ and mean y, and $\nu_n(A) = \nu(A)$ for all $A \subseteq (a_n, b_n)$;*

2. *$(\nu_n)_{n\geq n_0}$ is increasing in convex order with $\nu_n \to \nu$, and $b_n\nu_n(\{b_n\}) \to y - \overline{\nu}$.*

Proof. Suppose first that $\ell = -\infty$ or more generally that ν places no mass at ℓ. We look for a solution with $b_n = n$.

Fix $n_0 > \max\{y, (y-\overline{\nu})^{-1}\}$ and for $n \geq n_0$ let $v_n = F_\nu(n-)$. Define $H_n : [0, v_n] \mapsto \mathbb{R}$ via

$$H_n(w) = \int_w^{v_n} F_\nu^{-1}(u)du + n(w + 1 - v_n).$$

Then $H_n(0) = \int_0^{v_n} F^{-1}(u)du + n(1-v_n) \leq \int_0^1 F^{-1}(u)du = \overline{\nu} < y$, $H_n(v_n) = n > y$ and H_n is strictly increasing and continuous. Hence there exists a unique value $u_n > 0$ such that $H_n(u_n) = y$. Set $a_n = F_\nu^{-1}(u_n) > \ell$. Then $Z_n := F_\nu^{-1}(U)I_{\{u_n < U \leq v_n\}} + nI_{\{(U \leq u_n) \cup (U > v_n)\}}$ has mean y. Let ν_n be the law of Z_n. For $A \subseteq (a_n, b_n)$ we have $\nu_n(A) = \nu(A)$ and moreover $\nu_n([\ell, a_n)) = 0 = \nu_n((n, \infty])$. The measure ν_n has an atom at n of size $u_n + (1 - v_n)$ (and potentially an atom at a_n) and mean $\overline{\nu}$.

Now we argue that $H_{n+1}(u_n) > y$ and hence that $u_{n+1} < u_n$. We have

$$\begin{aligned} H_{n+1}(u_n) &= \int_{u_n}^{v_{n+1}} F_\nu^{-1}(u)du + (n+1)(u_n + 1 - v_{n+1}) \\ &= \int_{u_n}^{v_n} F_\nu^{-1}du + n(u_n + 1 - v_n) \\ &\qquad + \int_{v_n}^{v_{n+1}} F_\nu^{-1}(u)du - n(v_{n+1} - v_n) + (u_n + 1 - v_n) \\ &\geq y + u_n + (1 - v_n) \ > \ y. \end{aligned}$$

From the definition of u_n we have

$$y = \int x\nu_n(dx) = \int_{u_n}^{v_n} F_\nu^{-1}(u)du + n(1 + u_n - v_n)$$

and hence $\lim_n n(1+u_n-v_n)$ exists and is equal to $y - \int_{u_\infty}^1 F_\nu^{-1}(u)du$ where $u_\infty = \lim_n u_n$. In particular $\limsup nu_n < \infty$ and hence $u_\infty = 0$. Then $\lim_n n(1 + u_n - v_n) = y - \overline{\nu}$. Further, $(a_n)_{n \geq n_0}$ is a decreasing sequence with limit equal to the lower limit on the support of ν.

It remains to show that $(\nu_n)_{n \geq n_0}$ is increasing in convex order. This will follow if $\mathbb{E}[(z - Z_n)^+]$ is increasing in n for each z. For $z \leq n$,

$$\mathbb{E}[(z - Z_{n+1})^+] = \int_{u_{n+1}}^{F_\nu(z)} dw(z - F_\nu^{-1}(w)) > \int_{u_n}^{F_\nu(z)} dw(z - F_\nu^{-1}(w)) = \mathbb{E}[(z - Z_n)^+],$$

whereas for $z > n > y$, since $Z_n \leq n$,

$$\mathbb{E}[(z - Z_{n+1})^+] \geq \mathbb{E}[(z - Z_{n+1})] = z - y = \mathbb{E}[(z - Z_n)^+].$$

Now suppose that the target law has an atom at ℓ. In that case, since we require $a_n > \ell$ we need to extend the construction slightly.

Fix $n_1 > \max\{y, 4(y - \overline{\nu})^{-1}\}$ and for $n \geq n_1$ let $v_n = F_\nu(n-)$. Modify the definition of $H_n : [0, v_n] \mapsto \mathbb{R}$ to

$$H_n(w) = \int_w^{v_n} \max\{F_\nu^{-1}(u), \ell + 1/n\}du + n(w + 1 - v_n).$$

Then $H_n(0) \leq \int_0^1 [F^{-1}(u) + 1/n]du = \overline{\nu} + 1/n < y$, and as before there exists a unique value u_n such that $H_n(u_n) = y$. Set $a_n = \max\{F_\nu^{-1}(u_n), \ell + 1/n\} > \ell$. Then $Z_n := \max\{F_\nu^{-1}(U), \ell + 1/n\}I_{\{u_n < U \leq v_n\}} + nI_{\{(U \leq u_n) \cup (U > v_n)\}}$ has mean y. As before let ν_n be the law of Z_n.

We begin by finding a bound on u_n. Let n_2 be such that $\int_{v_{n_2}}^1 dw(F_\nu^{-1}(w) - n_2) < (y-\overline{\nu})/2$, and then this inequality holds for all $n \geq n_2$. We have, for $n \geq n_2$,

$$\begin{aligned} y &\geq \int_{u_n}^{v_n} F_\nu^{-1}(u)du + n(u_n + 1 - v_n) \\ &= \int_{u_n}^{1} F_\nu^{-1}(u)du + nu_n - \int_{v_n}^{1} [F_\nu^{-1}(u) - n]du \\ &> \overline{\nu} - \int_0^{u_n} F_\nu^{-1}(u)du + nu_n - \frac{y-\overline{\nu}}{2}. \end{aligned}$$

Note that $\Psi(w) = \int_0^w F_\nu^{-1}(u)du$ is convex and satisfies $\Psi(0) = 0$ and $\Psi(1) = \overline{\nu}$ and hence $\Psi(w) \leq \overline{\nu}w$. We conclude $y > \overline{\nu} + (n-\overline{\nu})u_n - \frac{y-\overline{\nu}}{2}$ and hence $u_n < \frac{(y-\overline{\nu})}{2(n-\overline{\nu})}$.

Let ϵ be such that $\int_0^\epsilon (y - F_\nu^{-1}(w))dw < \frac{y-\overline{\nu}}{4}$, and let n_3 be such that $\frac{y-\overline{\nu}}{2(n_3-\overline{\nu})} < \epsilon$. Then for $n \geq \max\{n_2, n_3\}$ we have $\int_0^{u_n} (y - F_\nu^{-1}(w))dw < \frac{y-\overline{\nu}}{4}$.

Now we argue that for $n \geq n_0 = \max\{n_1, n_2, n_3, n_4\}$ we have $H_{n+1}(u_n) > y$ and hence that $u_{n+1} < u_n$. This follows as in the previous case:

$$H_{n+1}(u_n) \geq \int_{u_n}^{v_{n+1}} F_\nu^{-1}(u)du + (n+1)(u_n + 1 - v_{n+1}) \geq y + u_n + (1-v_n) > y.$$

It follows that $(a_n)_{n\geq n_0}$ is a decreasing sequence. Further, from the definition of u_n we have

$$y = \int x\nu_n(dx) = \int_{u_n}^{v_n} \max\{F_\nu^{-1}(u), \ell + 1/n\}du + n(1 + u_n - v_n)$$

from which we conclude $\lim_n n(1 + u_n - v_n)$ exists and is equal to $y - \overline{\nu}$.

Finally, for $z \leq n$,

$$\begin{aligned} \mathbb{E}[(z - Z_{n+1})^+] &= \int_{u_{n+1}}^{F_\nu(z)} dw \left(z - \max\left\{F_\nu^{-1}(w)), \ell + \frac{1}{(n+1)}\right\}\right) \\ &> \int_{u_n}^{F_\nu(z)} dw \left(z - \max\left\{F_\nu^{-1}(w), \ell + \frac{1}{n}\right\}\right) = \mathbb{E}[(z - Z_n)^+], \end{aligned}$$

and it follows that $(\nu_n)_{n\geq n_0}$ is increasing in convex order. □

Recall the definition of $E_Y(y;\nu)$ in (2.4), or (2.5). Since we are assuming that $\nu \in L^1$ and $\overline{\nu} < y$, and since $\lim_{n\to\infty} \frac{q_y(y+n)}{n} = 2m((y,\infty))$, this simplifies to

$$E_Y(y;\nu) = \int q_y(z)\nu(dz) + 2(y-\overline{\nu})m((y,\infty)). \tag{3.1}$$

Theorem 3.6. *Suppose $\nu \in L^1$. Let σ be an embedding of ν. There exists an barrier B such that $\tau_B \wedge \sigma$ also has law ν and $\mathbb{E}[\tau_B \wedge \sigma] = E_Y(y;\nu)$.*

Proof. It only remains to cover the case where $Y_0 = y \neq \overline{\nu}$.

For each n, fix ν_n as above. From our study of the bounded, centred case we know there is a barrier B_n which we can assume contains $\{(t,x), x \leq a_n \text{ or } x \geq b_n \equiv n\}$ such that $Y_{\tau_{B_n}\wedge\sigma}$ has law ν_n. Moreover, exactly as in the proof of Theorem 3.4 (or more precisely, the proof of Lemma 4 in Monroe [15]), if $p > n$ then we may assume $B_p \subset B_n$. Then $\tau_{B_n} \uparrow \tau_\infty$ and $\tau_\infty \wedge \sigma$ embeds ν.

Finally we show that $\mathbb{E}[\sigma \wedge \tau_\infty] = E_Y(y;\nu)$.

Observe that q is convex and so $\lim_n q(n)/n$ exists in $(0,\infty]$. Then, as before $\mathbb{E}[\sigma \wedge \tau_\infty] = \lim \mathbb{E}[\sigma \wedge \tau_{B_n}] = \lim E_Y(y;\nu_n) = \lim \int q(z)\nu_n(dz)$ but in this case

$$\begin{aligned}
\int q(x)\nu_n(dx) &= \int_{u_n}^{v_n} q(\max\{F_\nu^{-1}(u),(\ell+1/n)\})du + q(n)(1-u_n-v_n) \\
&\to \int_0^1 q(F_\nu^{-1}(u))du + \lim_n \left\{ \frac{q(n)}{n} n(1+u_n-v_n) \right\} \\
&= \int q(x)\nu(dx) + (y-\overline{\nu}) \lim_n \left\{ \frac{q(n)}{n} \right\} \\
&= E_Y(y;\nu).
\end{aligned}$$

□

Proof of Theorem 2.4 in the case $\nu \in L^1$. If $E_Y(y;\nu) = \infty$ then since any embedding has $\mathbb{E}[\sigma] \geq \mathbb{E}[\sigma \wedge \tau_\infty] = E_Y(y;\nu)$ there are no integrable embeddings. Conversely, if $E_Y(y;\nu) < \infty$, then by Theorem 3.4 or Theorem 3.6 there exists an embedding $\tilde{\sigma}$ with $\mathbb{E}[\tilde{\sigma}] = E_Y(y;\nu)$.

Now suppose $E_Y(y;\nu) < \infty$ and σ is an embedding of ν.

Suppose σ is minimal. Choose ν_n as in the discussion before Theorem 3.4 or Theorem 3.6 as appropriate. In both of these theorems it was shown that we could choose a sequence of decreasing barriers B_n such that $\tau_{B_n} \wedge \sigma \to \tau_{B_\infty} \wedge \sigma$ and $\tau_{B_\infty} \wedge \sigma$ embeds ν. By minimality of σ, $\tau_{B_\infty} \wedge \sigma = \sigma$. Then, since $\tau_{B_n} \wedge \sigma$ is increasing,

$$\mathbb{E}[\sigma] = \mathbb{E}[\tau_{B_\infty} \wedge \sigma] = \lim_n \mathbb{E}[\tau_{B_n} \wedge \sigma] = \lim_n \int q(x)\nu_n(dx) = E_Y(y;\nu).$$

Conversely, if σ is not minimal then there is an embedding $\hat{\sigma}$ of μ with $\hat{\sigma} \leq \sigma$, $\mathbb{P}(\hat{\sigma} < \sigma) > 0$ and $\hat{\sigma}$ integrable. Then $\mathbb{E}[\sigma] > \mathbb{E}[\hat{\sigma}] \geq E(y;\nu)$. □

Example 3.7. *The following example shows that unlike in the Brownian case, in general integrability alone is not sufficient for minimality.*

Suppose the diffusion Y solves $dY_t = (1+Y_t^2)dW_t$ subject to $Y_0 = 0$. Let $\nu = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$ so that ν is uniform measure on $\{\pm 1\}$. Let $\hat{H} = H_1^Y \vee H_{-1}^Y$. Then $\hat{H}$ embeds ν and $\mathbb{E}[\hat{H}] < \infty$, but $\hat{H}$ is not minimal since $\hat{H} > H_1^Y \wedge H_{-1}^Y$ which is also an embedding of ν.

Example 3.8. *This example gives another circumstance in which integrability is not sufficient to guarantee minimality.*

Let Y be a time-homogeneous martingale diffusion on $I = [\ell, r]$ with $-\infty < \ell < y < r < \infty$. Suppose ℓ and r are exit boundaries and that $\mathbb{E}[H_{\ell,r}^Y] < \infty$. We take ℓ and r to be absorbing boundaries. (A simple example is obtained by taking Brownian motion started at y and absorbed at ℓ and r.) Let $\nu = \frac{(r-y)}{(r-\ell)}\delta_\ell + \frac{(y-\ell)}{(r-\ell)}\delta_r$. Then for $c > 0$, $H_{\ell,r}^Y + c$ is an integrable embedding which is not minimal.

However, examples of this type are degenerate and may easily be excluded by restricting the class of embeddings to those satisfying $\sigma \leq H_{\ell,r}^Y$.

Example 3.9. *Now we give an example which shows that minimality alone is not sufficient for integrability.*

Let Y be geometric Brownian motion so that Y solves $dY_t = Y_t dW_t$. Let Y have initial value $Y_0 = 1$. It is easy to see that for $a \in (0,1]$ we have

$$\mathbb{E}[H_a] = 2\int_a^\infty [(z \wedge 1) - a]\frac{dz}{z^2} = 2\log\left(\frac{1}{a}\right).$$

Let $\nu = \delta_0$. Then $\tau = \infty$ is the minimal stopping time that embeds ν in Y. Obviously τ is not integrable.

More generally, let ν be any probability measure on $(0,1)$ with $\int \log y \; \nu(dy) = -\infty$, and let Z be a random variable such that $\mathcal{L}(Z) \sim \nu$. Let the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be such that Z is $\mathcal{F}_0$-measurable, and let W be a $\mathbb{F}$-Brownian motion which is independent of Z.

Let $\tau = \inf\{u \geq 0 : Y_u = Z\}$. Then τ is an embedding of ν. Note that τ is a stopping time with respect to $\mathbb{F}$ but not with respect to the smaller filtration generated by Y alone. Moreover,

$$\mathbb{E}[\tau] = -2 \int \log z \; \nu(dz) = \infty.$$

Observe that $q_1(x) = \int_1^x \int_1^y \frac{2}{z^2} dz dy = 2(x-1) - 2\log(x)$. Hence $\lim_{x\to\infty} q_1(x)/x = 2$. Therefore, for any law ν on $(0,1)$, for a minimal embedding

$$\mathbb{E}[\tau] = 2\int (z-1)\nu(dz) - 2\int \log z \; \nu(dz) + 2(1-\overline{\nu}) = -2\int \log z \; \nu(dz).$$

We give another example of a minimal non-integrable embedding which does not require independent randomisation in the section on the Azéma-Yor stopping time.

Another feature of this example, is that Y is a martingale and yet it is easy to construct examples with $\overline{\nu} < y$ for which there is an integrable embedding. Hence integrability and minimality of τ is not sufficient for uniform integrability of $(Y_{t\wedge\tau})_{t\geq 0}$.

3.4 Alternative characterisations of E for the uncentred case

In the comments at the end of Section 2 we argued that in the non-centred case a natural family of embeddings was those which first involved waiting for the process to hit $\overline{\nu}$ and then to embed ν in Y started at $\overline{\nu}$. For a stopping rule τ as given in (2.2) we have from the analysis of the centred case that

$$\mathbb{E}[\tau] = \mathbb{E}^y[H_{\overline{\nu}}] + E_Y(\overline{\nu};\nu). \tag{3.2}$$

Now we want to show that the right hand side of (3.2) is equivalent to the expression given in (2.4).

More generally, for $v \in [\overline{\nu}, y]$ we could imagine waiting for the process to hit v and then using a minimal embedding time to embed ν in Y started at v. Then we find

$$\mathbb{E}[\tau] = \mathbb{E}^y[H_v] + E_Y(v;\nu). \tag{3.3}$$

We want to show that the right-hand-side of (3.3) does not depend on v.

Lemma 3.10. *Suppose $\overline{\nu} < y$ and define $G : [\overline{\nu}, y] \mapsto \mathbb{R}$ via*

$$G(v) = 2\int_v^\infty (y \wedge z - v) m(dz) + \int q_v(z)\nu(dz) + (v - \overline{\nu}) \lim_{n\uparrow\infty} \frac{q_v(v+n)}{n}$$

Then G does not depend on v. In particular, for all $v \in [\overline{\nu}, y]$, $E_Y(y,\nu) = \mathbb{E}^y[H_v] + E_Y(v;\nu)$. If this expression is finite for any (and then all) $v \in [\overline{\nu}, y]$ then for any embedding τ of ν we have that $\mathbb{E}[\tau] = \mathbb{E}^y[H_v] + E_Y(v;\nu)$.

Proof. For any u, v,

$$q_u(z) = q_u(v) + q_v(z) + q_u'(v)(z-v).$$

Then, with $u = \overline{\nu}$, $q_v(z) = q_{\overline{\nu}}(z) - q_{\overline{\nu}}(v) + q'_{\overline{\nu}}(v)(v-z)$ and for $v \in [\overline{\nu}, y]$

$$\begin{aligned} G(v) &= 2\int_v^y (z-v)m(dz) + 2(y-v)\int_y^\infty m(dz) + \int q_{\overline{\nu}}(z)\nu(dz) - q_{\overline{\nu}}(v) \\ &\quad + (v-\overline{\nu})q'_{\overline{\nu}}(v) + 2(v-\overline{\nu})\int_v^\infty m(dz) \\ &= 2(y-\overline{\nu})\int_y^\infty m(dz) + \int q_{\overline{\nu}}(z)\nu(dz) + 2\int_{\overline{\nu}}^y (z-\overline{\nu})m(dz) \end{aligned}$$

which does not depend on v. □

3.5 Non-integrable target laws

We have seen that if $\nu \in L^1$ then there exists an integrable embedding of ν if $\mathbb{E}^y[H_{\overline{\nu}}]$ and $\int q_{\overline{\nu}}(x)\nu(dx)$ are both finite. In this short section we argue that if $Y_0 = y \in (\ell, r)$ and $\nu \notin L^1$ then there does not exist an integrable embedding of ν.

Note first that $q = q_y$ is non-negative and convex, and hence $q(x) \geq \alpha|x-y| - \beta$ for some pair of finite positive constants α, β. Let T_n be a localising sequence for the local martingale $\{q(Y_{t\wedge\sigma}) - (t\wedge\sigma)\}_{t\geq 0}$. Then, by an argument similar to that in the proof of Lemma 3.1

$$\mathbb{E}[\sigma] = \lim_n \mathbb{E}[\sigma\wedge T_n] = \liminf \mathbb{E}[q(Y_{\sigma\wedge T_n})] \geq \mathbb{E}[\liminf q(Y_{\sigma\wedge T_n})] = \int q(z)\nu(dz) = \infty.$$

4 Diffusions started at entrance points

In the proofs of the main results we assumed that Y started at an interior point in (ℓ, r). Now we consider what happens if we start at a boundary point. The motivating example is a Bessel process in dimension 3 started at zero.

After a change of scale we may assume that we are working with a diffusion in natural scale. Then, if the boundary point is finite and an entrance point, it must also be an exit point (for terminology, see Borodin and Salminen [6, Section II.6]). We have assumed exit boundary points to be absorbing. It follows that an entrance point must be infinite; without loss of generality we assume that Y starts at $+\infty$ and that $I = [\ell, \infty]$ or $I = (\ell, \infty]$ where we may have $\ell = -\infty$.

So suppose that ∞ is an entrance-not-exit point. In particular, $\mathbb{E}^\infty[H_z] < \infty$ for some $z \in (\ell, \infty)$ or equivalently $\int^\infty zm(dz) < \infty$. We suppose the initial sigma algebra $\mathcal{F}_0$ is sufficiently rich as to include an independent, uniformly distributed random variable.

Define $E_Y(\infty, \nu)$ by

$$E_Y(\infty; \nu) := 2\int_\ell^\infty \nu(dx)\int_x^\infty m(dz)(z-x). \tag{4.1}$$

Theorem 4.1. *Suppose Y is a diffusion in natural scale and suppose $Y_0 = \infty$, where ∞ is an entrance-not-exit point.*

(i) There exists an integrable embedding of ν if and only if $E_Y(\infty; \nu)$ is finite.

(ii) Every minimal embedding σ of ν in Y has $\mathbb{E}[\sigma] = E_Y(\infty; \nu)$.

Remark 4.2. *Note that $E_Y(\infty; \nu)$ can be rewritten as*

$$E_Y(\infty; \nu) = 2\int_\ell^\infty m(dz)\int_\ell^z \nu(dx)(z-x)$$

It follows that if $\ell = -\infty$ and $\int_{-\infty}^0 |x|\nu(dx) = \infty$ then $E_Y(\infty; \nu) = \infty$.

However, it is easily possible to have $\nu \notin L^1$ and still have $E_Y(\infty;\nu) < \infty$ and the existence of integrable embeddings. For example, suppose Y solves $dY_t = Y_t^2 dB_t$ subject to $Y_0 = \infty$ and suppose ν is a measure on $(0,\infty)$ with $\int_0^\infty x\nu(dx) = \infty$ and $\int_0^\infty \nu(dx)/x^2 < \infty$, eg $\nu([x,\infty)) = x^{-1} \wedge 1$. Then $E_Y(\infty;\nu) = \int x^{-2}\nu(dx)/3 < \infty$ but $\nu \notin L^1$.

Remark 4.3. *Suppose that $\nu \in L^1$. Then as in Section 3.4 we can rewrite $E_Y(\infty;\nu)$ as*

$$E_Y(\infty;\nu) = 2\int_{\overline{\nu}}^{\infty} (y-\overline{\nu})m(dy) + \int q_{\overline{\nu}}(y)\nu(dy). \tag{4.2}$$

Occupation time arguments (Rogers and Williams [20, Section V.51]) imply that $2\int_{\overline{\nu}}^{\infty}(y-\overline{\nu})m(dy) = \mathbb{E}^\infty[H_{\overline{\nu}}^Y]$ and hence the right-hand-side of (4.2) has a clear interpretation as the sum of the expected time to hit the mean of the target law and the expected time to embed law ν in Y started at $\overline{\nu}$ using a minimal embedding. It follows that if $\nu \in L^1$ and there exists an integrable embedding of ν started at $\overline{\nu}$ then the stopping time 'run until Y hits the mean, and then use a minimal embedding to embed ν in Y started from the mean' is a minimal and integrable embedding.

Proof of Theorem 4.1. Here we prove the theorem in the case that $\nu \in L^1$. A proof in the case $\nu \notin L^1$ is given in the Appendix.

(i) Suppose first that $E_Y(\infty;\nu)$ is finite.

If $\nu \in L^1$ then we do not need $\mathcal{F}_0$ to be non-trivial. In this case both $\mathbb{E}^\infty[H_{\overline{\nu}}]$ and $\int q_{\overline{\nu}}(y)\nu(dy)$ are finite (since $E_Y(\infty;\nu)$ is). Then there exists a minimal, integrable, non-randomised embedding $\tau^{\overline{\nu},\nu}$ of ν in Y started at $\overline{\nu}$ and

$$\tau = H_{\overline{\nu}} + \tau^{\overline{\nu},\nu} \circ \Theta_{H_{\overline{\nu}}}$$

is an integrable embedding.

Now suppose there is an integrable embedding. The finiteness of $E_Y(\infty,\nu)$ is a corollary of the following lemma. Note that we do not need to assume $\nu \in L^1$.

Lemma 4.4. *Suppose $L > \ell$ and suppose $\tau \leq H_L$. Then τ is minimal for $\mathcal{L}(Y_\tau)$ in Y started at ∞ and $\mathbb{E}[\tau] = \mathbb{E}[q(Y_\tau)] = E_Y(\infty,\mathcal{L}(Y_\tau)) < \infty$.*

Suppose ρ is an embedding of ν. Then $\mathbb{E}[\rho] \geq E_Y(\infty,\nu)$.

Proof. The result is an analogue of Lemma 3.1, and the proof follows using similar ideas and the facts that q_∞ is bounded on $[L,\infty]$ and $E_Y(\infty,\nu) = \int q_\infty(y)\nu(dx)$ by definition. □

Return to the proof of Theorem 4.1(ii). If $E_Y(\infty,\nu) = \infty$ then there is nothing to prove. Suppose that $E_Y(\infty,\nu) < \infty$ and σ is minimal. We want to show that $\mathbb{E}[\sigma] = E_Y(\infty,\nu)$.

Let $\tilde{\sigma}_n = \max\{\sigma, H_n\}$ and let $\nu_n = \mathcal{L}(Y_{\tilde{\sigma}_n})$. Write $\tilde{\sigma}_n = H_n + \hat{\sigma}_n$ where $\hat{\sigma}_n = (\sigma - H_n)^+$. Set $\hat{\nu}_n = \mathcal{L}(Y^n_{\hat{\sigma}_n})$ where here the superscript reflects the fact that Y starts at n. Then $\hat{\nu}_n = \nu_n$.

We want to argue that $\overline{\nu}_n \leq n$ for sufficiently large n. To see this, first note that $\nu_n = \nu$ on $(-\infty,0)$ so that $\int_\ell^0 |x|\nu_n(dx) < \infty$ and $\overline{\nu}_n$ exists in $(-\infty,\infty]$, and second note that for $n \geq \overline{\nu}$ we have $\overline{\nu}_n \leq \overline{\nu} \leq n$.

To complete the proof of the theorem we need the following lemma.

Lemma 4.5. *For each n, $\hat{\sigma}_n$ is minimal for ν_n in Y started at n.*

Proof. Suppose $\hat{\rho}_n \leq \hat{\sigma}_n$ also embeds ν_n in Y started from n. If ρ is defined by

$$\rho = \begin{cases} \sigma & \sigma < H_n \\ H_n + \hat{\rho}_n \circ \Theta_{H_n} & \sigma \geq H_n \end{cases}$$

then $\rho \leq \sigma$. We show that $Y_\rho \sim Y_\sigma$; then by minimality of σ we conclude that $\rho = \sigma$. Hence $\hat{\rho}_n = \hat{\sigma}_n$ and $\hat{\sigma}_n$ is minimal as required.

By hypothesis, $\mathcal{L}(Y^n_{\hat{\sigma}_n}) = \mathcal{L}(Y^n_{\hat{\rho}_n})$. On $\sigma < H_n$ we have that $\hat{\sigma}_n = 0$ and hence $\hat{\rho}_n = 0$ and

$$\mathcal{L}(Y^n_{\hat{\sigma}_n}; \sigma < H_n) = \mathcal{L}(Y^n_{\hat{\rho}_n}; \sigma < H_n), \tag{4.3}$$

since in both cases this law is a point mass of size $\mathbb{P}(\sigma < H_n)$ at n. Then

$$\mathcal{L}(Y^n_{\hat{\sigma}_n}; \sigma < H_n) + \mathcal{L}(Y^n_{\hat{\sigma}_n}; \sigma \geq H_n) = \mathcal{L}(Y^n_{\hat{\sigma}_n}) = \mathcal{L}(Y^n_{\hat{\rho}_n}) = \mathcal{L}(Y^n_{\hat{\rho}_n}; \sigma < H_n) + \mathcal{L}(Y^n_{\hat{\rho}_n}; \sigma \geq H_n)$$

Then (4.3) means that we also have $\mathcal{L}(Y^n_{\hat{\sigma}_n}; \sigma \geq H_n) = \mathcal{L}(Y^n_{\hat{\rho}_n}; \sigma \geq H_n)$ and it follows that

$$\begin{aligned}
\mathcal{L}(Y_\rho) &= \mathcal{L}(Y_\sigma; \sigma < H_n) + \mathcal{L}(Y_{H_n + \hat{\rho}_n \circ \Theta_{H_n}}; \sigma \geq H_n) \\
&= \mathcal{L}(Y_\sigma; \sigma < H_n) + \mathcal{L}(Y^n_{\hat{\rho}_n}; \sigma \geq H_n) \\
&= \mathcal{L}(Y_\sigma; \sigma < H_n) + \mathcal{L}(Y^n_{\hat{\sigma}_n}; \sigma \geq H_n) \\
&= \mathcal{L}(Y_\sigma; \sigma < H_n) + \mathcal{L}(Y_{H_n + \hat{\sigma}_n \circ \Theta_{H_n}}; \sigma \geq H_n) = \mathcal{L}(Y_\sigma).
\end{aligned}$$

□

Return to the proof of Theorem 4.1. Since $\hat{\sigma}_n$ is minimal and $\overline{\nu}_n \leq \overline{\nu}$, we have that for $n \geq \overline{\nu}$, $\mathbb{E}^n[\hat{\sigma}_n] = E_Y(n, \nu_n) = \int q_n(x)\nu_n(dx) + 2(n - \overline{\nu}_n)m((n,\infty))$. Then

$$\begin{aligned}
\mathbb{E}^\infty[\sigma] &= \lim_n \mathbb{E}[(\sigma - H_n)^+] \\
&= 2\lim_n \left\{ \int_\ell^\infty \nu_n(dx) \int_n^x m(dz)(x-z) + (n - \overline{\nu}_n)m((n,\infty)) \right\}
\end{aligned} \tag{4.4}$$

Since ∞ is an entrance boundary $\int^\infty y m(dy) < \infty$ and hence $\lim_n nm((n,\infty) = 0$. In particular, $\lim_n (n - \overline{\nu}_n)m(n,\infty) \to 0$.

For the first term in (4.4), since $\nu_n = \nu$ on (ℓ, n),

$$\begin{aligned}
2\lim_n \left\{ \int_\ell^\infty \nu_n(dx) \int_n^x m(dz)(x-z) \right\} &\geq 2\lim_n \left\{ \int_\ell^n \nu(dx) \int_x^n m(dz)(z-x) \right\} \\
&= 2\int_\ell^\infty \nu(dx) \int_x^\infty m(dz)(z-x) \\
&= E_Y(\infty, \nu),
\end{aligned}$$

and conversely, since $\nu_n \leq \nu$ on (n, ∞),

$$\begin{aligned}
2\lim_n \left\{ \int_\ell^\infty \nu_n(dx) \int_n^x m(dz)(x-z) \right\} &\leq 2\lim_n \left\{ \int_\ell^\infty \nu(dx) \int_n^x m(dz)(x-z) \right\} \\
&= 2\int_\ell^\infty \nu(dx) \int_x^\infty m(dz)(z-x).
\end{aligned}$$

Hence $\mathbb{E}[\sigma] = E_Y(\infty, \nu) < \infty$.

□

5 Recovering results for general diffusions

Let $X = (X_t)_{t\geq 0}$ be a regular, time-homogeneous diffusion with state space I^X, started at $x \in int(I^X)$, and suppose that if X can reach an endpoint of I^X, then such an endpoint is absorbing. Then, see Rogers and Williams [20, Section V.44-47] or Borodin and Salminen [6], X has a scale function $s = s^X$ and speed measure m^X such that if $Y = (Y_t)_{t\geq 0}$ is given by $Y_t = s(X_t)$, then Y is a diffusion in natural scale with state space $I = s(I^X)$. For example, if X solves $dX = a(X_t)dW_t + b(X_t)dt$ then provided b/a^2 and $1/a^2$ are locally integrable,

$$s'(z) = \exp\left(-\int^z \frac{2b(v)}{a(v)^2}dv\right), \qquad m^X(dz) = \frac{dz}{a(z)^2 s'(z)}.$$

It follows that Y has speed measure

$$m(dy) = m^X(ds^{-1}(y)) = \frac{dy}{a(s^{-1}(y))^2 s'(s^{-1}(y))^2},$$

so that for $[L,R] \subset I$, $m((L,R)) = m^X((s^{-1}(L), s^{-1}(R)))$.

Let μ be a law on $\overline{I^X}$ and define $\nu = \mu \circ s^{-1}$ so that for a Borel subset of $\overline{I}$, $\nu(A) = \mu(s^{-1}(A))$. Then τ is an embedding of μ in X if and only if τ is an embedding of ν in Y. Moreover, the integrability of τ is also unaffected by a change of scale. Minimality is another property which is preserved under a change of scale.

We have that $\overline{\nu} := \int_I v\nu(dv) = \int_{I^X} s(z)\mu(dz)$ and $\int_I q_{s(x)}(v)\nu(dv) = \int_{I^X} q_{s(x)}(s(z))\mu(dz)$. Moreover, $q_y(z) = 2\int_y^z (z-w)m(dw) = 2\int_{s^{-1}(y)}^{s^{-1}(z)} (z - s(v))m^X(dv)$.

For definiteness suppose $s(x) > \overline{\nu}$, and denote by r the upper limit of I and by r^X the upper limit of I^X. Then $r = \infty$ and

$$\begin{aligned}
E_Y(s(x),\nu) &= \int_I q_{s(x)}(z)\nu(dz) + 2(s(x)-\overline{\nu})m((s(x),r)) \\
&= \int_{I^X} q_{s(x)}(s(z))\mu(dz) + 2(s(x)-\overline{\nu})m^X((x,r^X)) \\
&= 2\int_{I^X}\left\{\int_x^z (s(z)-s(v))m^X(dv)\right\}\mu(dz) + 2(s(x)-\overline{\nu})m^X(x,r^X)).
\end{aligned}$$

In general therefore, for $x \in int(I^X)$ set $E_X(x;\mu) = \infty$ if $\int_{I^X} |s(z)|\mu(dz) = \infty$ and otherwise

$$\begin{aligned}
E_X(x;\mu) &= 2\int_{I^X}\left\{\int_x^z (s(z)-s(v))m^X(dv)\right\}\mu(dz) \\
&\quad +2|s(x)-\overline{\nu}|\left(m^X((x,r^X))\mathcal{I}_{\{s(x)>\overline{\nu}\}} + m^X((l^X,x))\mathcal{I}_{\{s(x)<\overline{\nu}\}}\right).
\end{aligned} \tag{5.1}$$

As in the case for diffusions in natural scale, there is a second representation of E_X in terms of the expected value of first hitting time of the weighted mean of the target law together with the expected value of an embedding in a process started at the weighted mean, namely

$$E_X(x;\mu) = \mathbb{E}^x[H^X_{s^{-1}(\overline{\nu})}] + \int q_{\overline{\nu}}(s(z))\mu(dz). \tag{5.2}$$

Note that in this expression q is defined for the transformed process in natural scale.

Proof of Theorem 1.2. τ is minimal for μ in X started at x if and only if τ is minimal for ν in Y started at $y = s(x)$. Furthermore, τ is an integrable embedding of μ if and only if τ is an integrable embedding of ν. Then $\mathbb{E}[\tau] = E_Y(s(x);\nu) = E_X(x;\mu)$, where E_X is defined in either (5.1) or (5.2). □

Example 5.1. *Suppose P is a Bessel process of dimension 3, started at $p > 0$. Then the scale function is $s(x) = -x^{-1}$ and $I = (-\infty, 0)$. The speed measure is $m^P(dp) = p^2 dp$. There exists an embedding of μ in Y if and only if $\overline{\nu} \geq -p^{-1}$ where $\overline{\nu} = -\int_0^\infty x^{-1}\mu(dx)$. Further, there exists an integrable embedding of μ if and only if $E_P(p;\mu) < \infty$ where*

$$\begin{aligned} E_P(p;\mu) &= \int_0^\infty \mu(dz) 2 \int_p^z \left(\frac{1}{v} - \frac{1}{z}\right) v^2 dv + 2\left(\frac{1}{p} + \overline{\nu}\right)\frac{p^3}{3} \\ &= \frac{1}{3}\int_0^\infty z^2 \mu(dz) - \frac{p^2}{3} \end{aligned}$$

Example 5.2. *Suppose X is given by $X_t = aW_t + bt$ where $b > 0$ and W is standard Brownian motion, null at zero. Then $s(z) = -e^{-2bz/a^2}$ and $m^X(dz) = dx e^{2bz/a^2}/2b$. Set $\overline{\nu} = -\int_{\mathbb{R}} e^{-2bz/a^2}\mu(dz)$ and suppose $\overline{\nu} \in [-1, 0]$, else there is no embedding. Then $s^{-1}(\overline{\nu}) = -\frac{a^2}{2b}\log|\overline{\nu}|$ and $\exp(-\frac{2b}{a^2}s^{-1}(\overline{\nu})) = |\overline{\nu}|$. Hence*

$$\begin{aligned} \int \mu(dz) q_{\overline{\nu}}(s(z)) &= \int \mu(dz) 2 \int_{s^{-1}(\overline{\nu})}^z (s(z) - s(v)) m^X(dv) \\ &= \int \mu(dz) 2 \int_{s^{-1}(\overline{\nu})}^z (e^{-2bv/a^2} - e^{-2bz/a^2}) \frac{dv}{2b} e^{2bv/a^2} \\ &= \int \mu(dz) 2 \int_{s^{-1}(\overline{\nu})}^z (1 - e^{2b(v-z)/a^2}) \frac{dv}{2b} \\ &= \int \mu(dz) \left\{ \frac{z}{b} - \frac{s^{-1}(\overline{\nu})}{b} - \frac{a^2}{2b^2} + \frac{a^2}{2b^2} e^{2b(s^{-1}(\overline{\nu}) - z)/a^2} \right\} \\ &= \frac{1}{b}\int z\mu(dz) + \frac{a^2}{2b^2}\log|\overline{\nu}| - \frac{a^2}{2b^2} + \frac{a^2}{2b^2}\frac{1}{|\overline{\nu}|}\int e^{-2bz/a^2}\mu(dz) \\ &= \frac{1}{b}\int z\mu(dz) + \frac{a^2}{2b^2}\log|\overline{\nu}|. \end{aligned}$$

Suppose $X_0 = x$. For $w > x$ we have $\mathbb{E}^x[H_w^X] = (w - x)/b$. Then, using (5.2),

$$E_X(x;\mu) = \mathbb{E}^x[H^X_{s^{-1}(\overline{\nu})}] + \int q_{\overline{\nu}}(s(z))\mu(dz) = \frac{1}{b}\left(\int z\mu(dz) - x\right).$$

Recall from Lemma 3.1(ii) that every embedding of μ is minimal. Then, for drifting Brownian motion, every embedding of μ has the same expected value.

Remark 5.3. *Drifting Brownian motion was the subject of Grandits and Falkner [10], and the conclusion of the previous example is contained in their Proposition 2.2. Note that in the case $X_t = x + aB_t + bt$, if $\mathbb{E}[\tau] < \infty$ then $\mathbb{E}[X_\tau] - x = b\mathbb{E}[\tau]$. Hence, for an embedding τ of μ the result $\mathbb{E}[\tau] = E_X(x;\mu) = (\int z\mu(dz) - x)/b$ is not unexpected, and can be proved directly by other means.*

6 Minimality and Integrability of the Azéma-Yor embedding

Azéma and Yor [3, 2] (see also Rogers and Williams [20, Theorem VI.51.6] and Revuz and Yor [19, Theorem VI.5.4]), give an explicit construction of a solution of the SEP for Brownian motion. The original paper [3] assumes the target law is centred and square integrable, but the L^2 condition is replaced with a uniform integrability condition in [2], see also [19]. Azéma and Yor [3] also indicate how the results can be extended to diffusions, provided that the process is recurrent and provided that once the process has been transformed into natural scale, the mean of the target law is equal to the initial value of the diffusion.

The Azéma-Yor stopping time for a centred target law ν in Brownian motion W null at zero is

$$\tau^W_{AY,\nu} = \inf\{u : W_u \leq \beta_\nu(J^W_u)\}, \tag{6.1}$$

where J^W is the maximum process $J^W_u = \sup_{s\leq u} W_u$, and β_ν is the left-continuous inverse barycentre function, ie $\beta_\nu = b_\nu^{-1}$ where for a centred distribution η, $b_\eta(x) = \mathbb{E}^{Z\sim\eta}[Z|Z \geq x]$. The Azéma-Yor embedding has become one of the canonical solutions of the SEP because it does not involve independent randomisation and because it is possible to give an explicit form for the stopping time. Further, amongst uniformly integrable (or equivalently minimal) solutions of the SEP for Brownian motion, the Azéma-Yor solution has the property that it maximises the law of the stopped maximum, ie for all increasing functions H, $\mathbb{E}[H(J^W_\tau)]$ is maximised over minimal embeddings τ of ν in W by $\tau^W_{AY,\nu}$.

In the case where $\nu \in L^1$ but ν is not centred, Pedersen and Peskir [18] make the simple observation that we can embed ν by first running the Brownian motion until it hits $\overline{\nu}$ and then embedding ν in Brownian motion started at $\overline{\nu}$ using the classical centred Azéma-Yor embedding, ie they propose

$$\tau^W_{PP,\nu} = H^W_{\overline{\nu}} + \tau^W_{AY,\nu} \circ \Theta_{H^W_{\overline{\nu}}}.$$

However, if the Brownian motion is null at zero, and $\overline{\nu} < 0$, then the embedding $\tau_{PP,\nu}$ no longer maximises the law of the stopped maximum. Instead Cox and Hobson [8] introduce an alternative modificiation of the Azéma-Yor stopping time which does maximise the law of the stopped maximum, and it is this embedding which we will study here. In fact the expected value of any embedding of the form $H^Y_{\overline{\nu}} + \tau^{\overline{\nu},\nu} \circ \Theta_{H^Y_{\overline{\nu}}}$ can be found very easily, and our aim here is to analyse an embedding which is not of this form.

Suppose $W_0 = w$ and $\nu \in L^1$. Define $D_\nu(x) = \mathbb{E}^{Z\sim\nu}[(Z - x)^+] + (w - \overline{\nu})^+$ and for $z \geq w$ set

$$\beta_\nu(z) = \arg\inf_{v<z} \left\{ \frac{D_\nu(v)}{z - v} \right\}. \tag{6.2}$$

(Here the $\arg\inf$ may not be uniquely defined, but we can make the choice of β_ν unique by adding a left-continuity requirement.) Then the Cox-Hobson extension of the Azéma-Yor embedding is to set

$$\tau^W_{CH,\nu} = \inf\{u : W_u \leq \beta_\nu(J^W_u)\}. \tag{6.3}$$

Note that if $\overline{\nu} \geq w$, then for $z \in [w, \overline{\nu}]$ we have $\beta_\nu(z) = -\infty$. In this case the Cox-Hobson and Pedersen-Peskir embeddings are identical. However, if $\overline{\nu} < w$ then the Cox-Hobson and Pedersen-Peskir embeddings are distinct.

To ease the exposition we assume that ν has a density ρ. (The general case can be recovered by approximation, or by taking careful consideration of atoms.) Then $b = \beta_\nu^{-1}$ solves

$$(b(y) - y)\nu((y, \infty)) = D_\nu(y), \tag{6.4}$$

b is differentiable and $\nu((y, \infty))b'(y) = (b(y) - y)\rho(y)$. Then, writing τ for $\tau^W_{CH,\nu}$ and $L(\nu)$

for the lower limit of the support of ν and using excursion-theoretic arguments,

$$\begin{aligned}
\mathbb{P}(W_\tau > y) = \mathbb{P}(J^W_\tau > b(y)) &= \exp\left(-\int_w^{b(y)} \frac{dz}{z-\beta(z)}\right) \\
&= \exp\left(-\int_{w\vee\overline{\nu}}^{b(y)} \frac{dz}{z-\beta(z)}\right) \\
&= \exp\left(-\int_{L(\nu)}^{y} \frac{b'(v)}{b(v)-v}dv\right) \\
&= \exp\left(-\int_{L(\nu)}^{y} \frac{\rho(v)}{\nu((v,\infty))}dv\right) = \nu((y,\infty))
\end{aligned}$$

and hence $\tau^W_{CH,\nu}$ is an embedding of ν.

Cox and Hobson [9] prove that the embedding in (6.3) is minimal. A bi-product of the subsequent arguments in this section is a proof of minimality by different means. Note that this is only relevant in the case $I = \mathbb{R}$, else every embedding is minimal.

Let Y be a regular diffusion in natural scale. Then by the Dambis-Dubins-Schwarz theorem Y can be written as a time-change of Brownian motion: $Y_t = W_{[Y]_t}$ for some Brownian motion (on a filtration and probability space constructed from the original space supporting Y). Then if we set $Q = [Y]^{-1}$ we have $W_t = Y_{Q_t}$. Conversely, let W be Brownian motion and let $(L^W_t(z))_{t\geq 0, z\in\mathbb{R}}$ be its family of local times. Given a measure m on I (with a strictly positive density with respect to Lebesgue measure), set $A_s = \int_I m(dz)L^W_s(z)$. Then A is strictly increasing and continuous (at least until W hits an endpoint of I) and we can define an inverse $\Gamma = A^{-1}$. Finally set $Y_t = W_{\Gamma_t}$; then Y is a diffusion in natural scale with speed measure m.

It follows that if τ is a solution of the SEP for ν in W then Q_τ is a solution of the SEP for ν in Y. Similarly, if σ is the solution of the SEP in Y, then Γ_σ is a solution of the SEP in W. Hence there is a one-to-one correspondence between solutions of the SEP for ν in W and solutions for ν in Y.

Recall that we are supposing that $\nu \in L^1$. (Note that if $\nu \notin L^1$ then it is not possible to define $D_\nu(\cdot)$, and the Azéma-Yor solution is not defined.) Suppose also that $w > \overline{\nu}$, which is the interesting case in which the Pedersen-Peskir and Cox-Hobson embeddings are distinct. By analogy with (6.3) define

$$\tau^Y_{CH,\nu} = \inf\{u : Y_u \leq \beta_\nu(J^Y_u)\} \tag{6.5}$$

where β_ν is as defined in (6.2). Then $\tau = \tau^Y_{CH,\nu}$ inherits the embedding property from $\tau^W_{CH,\nu}$ and is a solution of the SEP for ν in Y.

Now consider the question of minimality. It is clear that $\tau^W_{CH,\nu}$ is minimal for ν in W if and only if τ is minimal for ν in Y. If $\overline{\nu} \neq y$ then $\tau^W_{CH,\nu}$ is not integrable, but τ may be integrable. Further, if τ is integrable for ν in Y started at w and if $E_Y(w;\nu) < \infty$ then τ is minimal if and only if $\mathbb{E}[\tau] = E_Y(w;\nu)$. In particular, if we choose the diffusion Y so that its speed measure satisfies $m(\mathbb{R}) < \infty$, then necessarily $E_Y(w;\nu) < \infty$ (recall $\nu \in L^1$). The minimality of τ for ν in Y and hence the minimality of $\tau^W_{AY,\nu}$ will follow if we can show $\mathbb{E}[\tau] = E_Y(w;\nu)$.

We have, (recall $w > \overline{\nu}$),

$$\begin{aligned}
\mathbb{E}[\tau] &= \int_w^\infty dz \mathbb{P}(J_\tau^Y \geq z) \int_{\beta(z)}^z \frac{2(x-\beta(z))}{z-\beta(z)} m(dx) \\
&= 2\int_{\mathbb{R}} \frac{b'(y)}{(b(y)-y)} dy \mathbb{P}(Y_\tau \geq y) \int_y^{b(y)} (x-y)m(dx) \\
&= 2\int_{\mathbb{R}} \rho(y)dy \int_y^{b(y)} (x-y)m(dx) \\
&= 2\int_{-\infty}^w m(dx) \int_{-\infty}^x (x-y)\rho(y)dy + 2\int_w^\infty m(dx) \int_{\beta(x)}^x (x-y)\rho(y)dy.
\end{aligned}$$

Here we use excursion theory and the fact that

$$\mathbb{E}^x[H_{a,b}^Y] = 2\int_a^b (x \wedge z - a)(b - x \vee z)m(dz) \qquad a < x < b$$

for the first line (see also Pedersen and Peskir [17, Theorem 4.1]), $(J_\tau^Y \geq z) = (Y_\tau \geq \beta(z))$ for the second line, $b'(y) = \rho(y)(b(y)-y)/\nu((y,\infty))$ almost everywhere for the third, and the fact that $b(y) \geq w$ for the final line.

Observe that

$$2\int_{-\infty}^w m(dx) \int_{-\infty}^x (x-y)\nu(dy) = 2\int_{-\infty}^w \nu(dy) \int_y^w (x-y)m(dx) = \int_{-\infty}^w \nu(dy) q_w(y).$$

Note that it is no longer true that $b = b_\nu = \beta_\nu^{-1}$ satisfies $b(y) = \mathbb{E}^{Y \sim \nu}[Y | Y \geq y]$ but rather $b(y) = \{(w-\overline{\nu}) + \int_y^\infty z\nu(dz)\}/(\nu(y,\infty))$ and then $(x-\beta(x))\int_{\beta(x)}^\infty \nu(dz) = w - \overline{\nu} + \int_{\beta(x)}^\infty (z-\beta(x))\nu(dz)$. Thus

$$\int_{\beta(x)}^x (x-y)\nu(dy) = \int_{\beta(x)}^\infty (x-y)\nu(dy) + \int_x^\infty (y-x)\nu(dy) = (w-\overline{\nu}) + \int_x^\infty (y-x)\nu(dy),$$

and

$$2\int_w^\infty m(dx) \int_{\beta(x)}^x (x-y)\nu(dy) = 2(w-\overline{\nu})m((w,\infty)) + \int_w^\infty q_w(y)\nu(dy).$$

Finally then,

$$\mathbb{E}[\tau] = 2(w-\overline{\nu})m((w,\infty)) + \int q_w(y)\nu(dy) = E_Y(w;\nu)$$

and hence τ and $\tau_{CH,\nu}^W$ given in (6.3) are minimal.

6.1 An example

In this example we suppose Y is a non-negative, regular, local-martingale diffusion started at 1 with state space unbounded above and absorbed at zero (if Y can hit zero in finite time, else Y is assumed to be transient to zero). We suppose further that ν is given by $\nu((y,\infty)) = (1+\theta y)^{-\phi}$ with $\theta, \phi > 0$ and $\phi \geq 1 + 1/\theta$. If $\phi = 1 + 1/\theta$ then $\overline{\nu} = 1$, otherwise if $\phi > 1 + 1/\theta$ then $\overline{\nu} < 1$. (Note that if $\phi < 1 + 1/\theta$, then $\overline{\nu} > 1$ and there is no embedding of ν in Y.)

Our first goal is to find the function β_ν in the Cox-Hobson extension of the Azéma-Yor embedding and the associated stopping times. In fact we find a family of solutions parameterised by $\psi \in [\overline{\nu}, 1]$ for which the stopping time with parameter ψ corresponds to running Y until it hits ψ and then embedding ν in Y started at ψ using the Cox-Hobson embedding. In particular this stopping time can be written as

$$H^Y_\psi + \tau^\psi \circ \Theta_{H^Y_\psi}$$

where

$$\tau^\psi = \inf\{u \geq 0; Y^\psi_u \leq \beta_{\nu,\psi}(J^{Y^\psi}_u)\}$$

and Y^ψ satisfies $Y^\psi_0 = \psi$. Here, for $\psi \in [\overline{\nu}, 1]$, $D_{\nu,\psi}(z) = \mathbb{E}^{Z\sim\nu}[(Z-x)^+] + (\psi - \overline{\nu})$ is given by

$$D_{\nu,\psi}(z) = \psi - \frac{1}{\theta(\phi-1)}\left\{1 - (1+\theta y)^{-(\phi-1)}\right\}$$

and $b = \beta^{-1}_{\nu,\psi}$ given by (6.4) has expression

$$b(y) = (1+\theta y)^\phi \left(\psi - \frac{1}{\theta(\phi-1)}\right) + \frac{\phi y}{\phi - 1} + \frac{1}{\theta(\phi-1)}.$$

Now suppose $m(dy) = y^{-2c}dy$ (with $c \in (0,\infty)\backslash\{1/2, 1\}$) so that Y solves $dY = Y^c dW$. Then $q = q_1$ is given by $q(x) = \frac{x^{2-2c}-1}{(1-c)(1-2c)} - \frac{2(x-1)}{(1-2c)}$. We have

$$E_Y(1;\nu) = \int_0^\infty q(y)\nu(dy) + 2(1-\overline{\nu})m((1,\infty))$$

Suppose $\phi > 1 + 1/\theta$. Then $\overline{\nu} < 1$ and there exists an integrable embedding of ν if and only if each of the three integrals

$$\int^\infty x^{-2c}dx, \qquad \int^\infty x^{2-2c}x^{-(\phi+1)}dx, \qquad \int_0 x^{2-2c}dx$$

is finite or equivalently $c > 1/2$, $c > 1 - \phi/2$ and $c < 3/2$. However, since $\phi \geq 1 + 1/\theta > 1$ this reduces to $1/2 < c < 3/2$.

If $\phi = 1 + 1/\theta$ then there is no requirement for $m((1,\infty))$ to be finite, the condition $c > 1/2$ is not needed and there exists an integrable embedding of ν if and only if $1 - \phi/2 < c < 3/2$.

These statements are consistent with the case $c = 0$ of absorbing Brownian motion. Then ν can be embedded in integrable time if and only if $\overline{\nu} = 1$ and $\nu \in L^2$, or equivalently $\phi = 1 + 1/\theta$ and $\phi > 2$.

6.2 An example of Pedersen and Peskir

Pedersen and Peskir [18] give the expected time for a Bessel process to fall below a constant multiple of the value of its maximum, ie they find $\mathbb{E}[\tau^P_{AY}]$ where $\tau^P_{AY} = \inf\{u > 0 : P_u \leq \lambda J^P_u\}$ and $\lambda < 1$. They find the answer by solving a differential equation subject to boundary conditions and a minimality principle. We can recover their result directly using our methods.

Let P be a Bessel process of dimension $\alpha \neq 2$, started at 1. Then $Y = P^{2-\alpha}$ is a diffusion in natural scale. Then P solves $dY_t = (2-\alpha)Y^b_t dW_t$ where $b = (1-\alpha)/(2-\alpha)$. Then $m(dy) = (2-\alpha)^{-2}y^{-2b}dy$ and

$$q_1(y) = \frac{1}{(2-\alpha)^2}\left[\frac{y^{2(1-b)}}{(1-b)(1-2b)} + \frac{1}{1-b} - \frac{2y}{1-2b}\right].$$

Suppose first $\alpha < 2$. We find, with $J_u = J^Y_u = \sup_{s\leq u} Y_u$,

$$\tau^P_{AY} = \inf\{u > 0; Y^{1/(2-\alpha)}_u \leq \lambda J^{1/(2-\alpha)}_u\} = \inf\{u > 0; Y_u \leq \gamma J_u\} =: \tau^\gamma$$

where $\gamma = \lambda^{2-\alpha}$. Then, by excursion theoretic arguments, see for example Rogers and Williams [20, Section VI.51], for $y \geq \gamma$,

$$\mathbb{P}(Y_{\tau^\gamma} \geq y) = \mathbb{P}(J_{\tau^\gamma} \geq y/\gamma) = \exp\left(- \int_1^{y/\gamma} \frac{dj}{(j - \gamma j)} \right) = (y/\gamma)^{-1/(1-\gamma)}.$$

Then, if $\nu = \mathcal{L}(Y_{\tau^\gamma})$ we have $\overline{\nu} = 1$ and

$$\mathbb{E}[\tau^\gamma] = \int_\gamma^\infty q_1(y)\nu(dy) = \frac{\lambda^\alpha(2-\alpha)}{\alpha(2-\alpha\lambda^{\alpha-2})} - \frac{1}{\alpha}$$

provided $\alpha\lambda^{\alpha-2} < 2$, and otherwise τ^γ is not integrable.

If $\alpha > 2$ then set $Y = -P^{2-\alpha}$. Then $\tau^P_{AY} = \inf\{u > 0 : Y_u \leq \gamma J_u\} =: \tau^\gamma$ where $\gamma = \lambda^{2-\alpha} > 1$. Then for $y \in (-\gamma, 0)$, $\mathbb{P}(Y_{\tau^\gamma} \geq y) = (|y|/\gamma)^{1/(\gamma-1)}$. Again we find that $\nu \sim \mathcal{L}(Y_{\tau^\gamma})$ has unit mean and

$$\mathbb{E}[\tau^\gamma] = \frac{\lambda^\alpha(\alpha-2)}{\alpha(\alpha\lambda^{\alpha-2}-2)} - \frac{1}{\alpha},$$

provided $\alpha\lambda^{\alpha-2} > 2$, else τ^γ is not integrable.

Finally, if $\alpha = 2$, we set $Y = \log P$ and then $\tau^P_{AY} = \inf\{u > 0 : Y_u \leq J_u - \gamma\} =: \tau^\gamma$ where $\gamma = -\log\lambda > 0$. Then, for $y \geq -\gamma$, $\mathbb{P}(Y_\tau \geq y) = e^{-(y/\gamma)-1}$. Further, $dY_t = e^{-Y_t}dB_t$ and if $P_0 = 1$ then $Y_0 = 0$. Then $m(dy) = e^{2y}dy$ and $q_0(y) = \{e^{2y} - 2y - 1\}/2$. Hence

$$\mathbb{E}[\tau^\gamma] = \int_{-\gamma}^\infty q_0(y)\nu(dy) = \int_{-\gamma}^\infty \frac{e^{2y-(y/\gamma)-1}}{2\gamma}dy - \frac{1}{2} = \frac{\lambda^2}{2+4\log\lambda} - \frac{1}{2}$$

provided $\lambda > e^{-1/2}$, and otherwise τ^γ is not integrable.

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A Proof of Theorem 4.1 when the target measure is not integrable

The proof of the theorem in the case $\nu \notin L^1$ will follow from the results of this section, and especially Lemma A.1 and Corollary A.9.

We begin with the forward implication in (i). Since Lemma 4.4 does not require $\nu \in L^1$, Theorem 4.1(i) is then proved in the general case. When we discuss Theorem 4.1(ii) we will use two approaches. In the first approach we give simple sufficient conditions such that $\overline{\nu}_n \leq n$, whence we can deduce the result using the reasoning in the main text. The second approach works in general, but needs a new style of argument.

Lemma A.1. *If $\nu \notin L^1$ but $E_Y(\infty, \nu) < \infty$, then there exists an integrable embedding.*

Proof. By assumption $\mathcal{F}_0$ is sufficiently rich as to include a uniform random variable. Then there exists a random variable Z with law ν and setting $\sigma = H^Y_Z$, where $H^Y_Z = \inf\{u \geq 0; Y_u \leq Z\}$ is the first hitting time by Y of the random level Z, we have $Y_\sigma \sim \nu$ and

$$\mathbb{E}[\sigma] = \int \nu(dz)\mathbb{E}^\infty[H^Y_z] = 2\int \nu(dz)\int_z^\infty (y-z)m(dy) = E_Y(\infty;\nu).$$

□

Remark A.2. *Note that if ν includes an atom at ∞ then independent randomisation at $t = 0$ will always be necessary to construct an embedding. However, if ν has no atom at infinity, then if $b_\nu^\infty(x) := \int_{[\ell,x]} y\nu(dy)/\int_{[\ell,x]}\nu(dy)$ (with $b_\nu^\infty(x) = \ell$ when the denominator in this expression is zero), if β_ν^∞ is the right-continuous inverse of b_ν^∞ and if $\tau := \inf\{u > 0 : Y_u \geq \beta_\nu^\infty(\inf_{s\leq u} Y_s)\}$ then τ is a non-randomised, minimal embedding of ν. This construction is an analogue of the Azéma-Yor embedding, adapted to the current context.*

Recall the definition of $\hat{\sigma}_n$ and law ν_n from the proof of Theorem 4.1.

Lemma A.3. *Suppose there exists an integrable embedding of ν. Then $\int_\ell^0 |x|\nu(dx) < \infty$ and hence $\overline{\nu}_n$ exists in $(-\infty, \infty]$.*

Proof. We only need consider the case $\ell = -\infty$. By Lemma 4.4, $E_Y(\infty, \nu) < \infty$. Since $q_\infty(x) \geq Cx^-$ for all $x < 0$ for some $C > 0$, the first part of the lemma follows.

The final statement follows from the fact that $\nu_n = \nu$ on $(-\infty, 0)$. □

Recall the setting that Y is a regular diffusion in natural scale with state space an interval with endpoints $\ell \geq -\infty$ and ∞ and speed measure m, started at the entrance boundary ∞. Hence $\int^\infty xm(dx) < \infty$ and therefore also $\int^\infty m(dx) < \infty$.

Lemma A.4. *Suppose $\ell > -\infty$ or $m(\mathbb{R}) = \infty$. Then Y started from n is a supermartingale.*

Proof. If the local martingale Y is bounded below then it follows from Fatou's Lemma that it is a supermartingale. Otherwise, since ∞ is an entrance boundary, $m(\mathbb{R}^+) < \infty$ and we must have $m(\mathbb{R}^-) = \infty$. Then the fact that Y is a supermartingale follows from Kotani [14], and more especially Theorem 2 of Gushchin et al [11]. □

Proposition A.5. *Suppose $\ell > -\infty$ or $m(\mathbb{R}) = \infty$. If there exists an integrable embedding of ν in Y then σ is minimal for ν in Y if and only if $\mathbb{E}[\sigma] = E_Y(\infty, \nu)$.*

Proof. The proof follows the proof in the case $\nu \in L^1$, except that we use the supermartingale property of Y to conclude that $\overline{\nu}_n \leq n$ for each n. □

Now we turn to the general case. Our proof is inspired by the ideas of Root [21] and Monroe [15], but is completely different in character. The proof uses independent randomisation.

The idea in Root [21] and Monroe [15] is to construct a barrier B such that $\sigma \wedge \tau_B$ has certain integrability properties and has the same law as Y_σ. Then if σ is minimal we deduce that σ has those same integrability properties. Our aim is to construct a random level Z, independent of Y and σ, such the randomized stopping time $\sigma \wedge H_Z^Y$ is minimal, has the same law as σ, and has expectation $E_Y(\infty, \nu)$. Then if σ is minimal we can conclude that $\mathbb{E}[\sigma] = E_Y(\infty, \nu)$.

Lemma A.6. *Suppose ν has finite support contained in $(\ell, \infty]$, and that σ is an embedding of ν. Then there exists a $\mathcal{F}_0$-measurable random variable Z, taking values in the support of ν, such that $\sigma \wedge H_Z^Y$ is minimal, $Y_{\sigma\wedge H_Z^Y}$ has law ν and $\mathbb{E}[\sigma \wedge H_Z^Y] = E_Y(\infty, \nu)$.*

Proof. Suppose ν has support $\{x_{k+1}, x_k, \dots x_1\}$ with $\ell < x_{k+1} < x_k < \dots < x_1 \leq \infty$. Let $g = (g_1, \dots g_k)$ be a vector in the non-negative orthant $\mathbb{R}_+^k$. Set $g_{k+1} = \infty$. Let Z_g be a random variable such that $\mathbb{P}(Z_g < z) = \exp(-\sum_{j:x_j\geq z} g_j)$. Then Z_g is a random variable with support contained in the support of ν and such that $\mathbb{P}(Z_g = x_j) = e^{-\sum_{i<j} g_i}(1 - e^{-g_j})$.

Let $\mathcal{G}_\nu$ be the set of vectors g such that $\mathbb{P}^\infty(Y_{\sigma\wedge H^Y_{Z_g}} = x_j) \le \nu(\{x_j\})$ for $1 \le j \le k$. (Note there is no restriction on the mass at x_{k+1}.) Clearly the zero vector in $\mathbb{R}^k_+$ is an element of $\mathcal{G}_\nu$. Let $\nu_g = \mathcal{L}(Y_{\sigma\wedge H^Y_{Z_g}})$; then for $1 \le j \le k+1$,

$$\nu_g(\{x_j\}) = \mathbb{E}[\exp(-\sum_{i:H^Y_{x_i}<\sigma} g_i); Y_\sigma = x_j] + \mathbb{P}(H_{x_j} \le \sigma)e^{-\sum_{i<j} g_i}(1-e^{-g_j}).$$

Suppose $\hat{g}$ and $\tilde{g}$ are both elements of $\mathcal{G}_\nu$. We show that the componentwise maximum $g = \hat{g} \vee \tilde{g}$ is also an element of $\mathcal{G}_\nu$. Fix j with $1 \le j \le k$. Without loss of generality $g_j = \hat{g}_j \ge \tilde{g}_j$. Then, since $(1-e^{-g_j}) = (1-e^{-\hat{g}_j})$, and $e^{-g_{j'}} \le e^{-\hat{g}_{j'}}$ for all j',

$$\nu_g(\{x_j\}) \le \mathbb{E}[\exp(-\sum_{i:H^Y_{x_i}<\sigma} \hat{g}_i); Y_\sigma = x_j] + \mathbb{P}(H_{x_j} < \sigma)e^{-\sum_{i<j} \hat{g}_i}(1-e^{-\hat{g}_j}) = \nu(\{x_j\})$$

Hence $g \in \mathcal{G}_\nu$.

It follows that $\mathcal{G}_\nu$ has a maximal element g^*. It must be the case that $\nu_{g^*} = \nu$. If not, then there must be some paths which reach x_{k+1}, and increasing g_j strictly increases $\nu_g(\{x_j\})$ whilst at the same time not increasing $\nu_g(\{x_{j'}\})$ for any $j' \neq j$, contradicting the maximality of g^*. □

Suppose $x_{k+2} < x_{k+1}$ and let $\tilde{\nu}$ be a measure on $\{x_{k+2}, x_{k+1}, \dots, x_1\}$ with $\tilde{\nu}(\{x_j\}) = \nu(\{x_j\})$ for $1 \le j \le k$. (In other words $\tilde{\nu}$ is obtained from ν by splitting the mass at the lowest point, and moving some of it to the even lower point x_{k+2}.) If $\tilde{g} = (\tilde{g}_1, \dots, \tilde{g}_{k+1}) \in \mathcal{G}_{\tilde{\nu}}$ then it follows that $(\tilde{g}_1, \dots, \tilde{g}_k) \in \mathcal{G}_\nu$: increasing $\tilde{g}_{k+1}$ to infinity can only decrease the probability of stopping at x_j for $j \le k$. Hence, if Z_ν and $Z_{\tilde{\nu}}$ are the random variables arising in Lemma A.6 for ν and $\tilde{\nu}$ respectively, then $\mathbb{P}(Z_\nu \le z) \le \mathbb{P}(Z_{\tilde{\nu}} \le z)$ for all z. Further there is a coupling such that $Z_{\tilde{\nu}} \le Z_\nu$: we simply take $Z_\nu = F^{-1}_{Z_\nu}(U)$ and $Z_{\tilde{\nu}} = F^{-1}_{Z_{\tilde{\nu}}}(U)$ for the same uniform random variable U in each case. Then also $H^Y_{Z_\nu} \le H^Y_{Z_{\tilde{\nu}}}$, almost surely.

Lemma A.7. *Suppose ν has support on the interval $[L, \infty]$, with $L > \ell$ and that σ is an embedding of ν. Then there exists a $\mathcal{F}_0$-measurable random variable Z taking values in $[L, \infty]$ such that $\sigma \wedge H^Y_Z$ is minimal, $Y_{\sigma\wedge H^Y_Z}$ has law ν and $\mathbb{E}[\sigma \wedge H^Y_Z] = E_Y(\infty, \nu)$.*

Proof. Without loss of generality we suppose L is a dyadic rational of the form $L = k_L 2^{-\kappa_L}$ for $k_L \in \mathbb{Z}^-$ and $\kappa_L \in \mathbb{Z}^+$.

Let $J \ge \kappa_L$ be a positive integer. Let ν^J be given by $\nu^J(\{J\}) = \nu([J, \infty])$ and $\nu^J(\{r2^{-J}\}) = \nu([r2^{-J}, (r+1)2^{-J}))$ for $L2^J \le r \le J2^J - 1$.

Then ν^J is a distribution on $\mathcal{S}^J = \{r2^{-J}; L2^J \le r \le J2^J\}$, and if $\sigma^J = \inf\{u \ge \sigma : Y_u \in \mathcal{S}^J \text{ and } Y_u \le Y_\sigma\}$ then $\nu^J = \mathcal{L}(Y_{\sigma^J})$.

From Lemma A.6 there exists a random variable Z^J such that $\mathcal{L}(Y_{\sigma^J \wedge H_{Z^J}}) = \nu^J$.

Let F^J denote the cumulative distribution function of Z^J; then by the Helly Selection Theorem (Billingsley [5, Theorem 25.9]) down a subsequence F^J converges at continuity points of the limit to an increasing, right-continuous function F, which we may consider as the distribution function of an extended random variable (ie. one taking values in $[-\infty, \infty]$, or in this case $[L, \infty]$).

We may assume $Z^J = (F^J)^{-1}(U)$ and then we have $Z^J \to Z$ almost surely. On $Z = \infty$ we set $H^Y_Z = 0$. With this convention (and taking as implicit the qualifier almost surely) $H^Y_{Z^J} \to H^Y_Z$ and $\sigma^J \downarrow \sigma$. Then $\sigma^J \wedge H^Y_{Z^J} \to \sigma \wedge H^Y_Z$ and $Y_{\sigma^J \wedge H^Y_{Z^J}} \to Y_{\sigma\wedge H^Y_Z}$. Hence

$$\mathcal{L}(Y_{\sigma\wedge H^Y_Z}) = \lim_J \nu^J = \nu.$$

Since $Z \ge L$ we have that $\sigma \wedge H^Y_Z \le H^Y_L$ and, by Lemma 4.4, $\sigma \wedge H^Y_Z$ is minimal and satisfies $\mathbb{E}[\sigma \wedge H^Y_Z] = E_Y(\infty, \nu)$. □

For L of the form $L = k_L 2^{-\kappa_L}$ as in the lemma, let $\sigma_L = \inf\{u \geq \sigma : Y_u \geq L\}$ and let $\nu_L = \mathcal{L}(Y_{\sigma_L})$. Then ν_L is the law of $X^L = \max\{X, L\}$ where X has law ν. Using a superscript to denote the fineness of the partition, and a subscript to denote the lower bound on the distribution, let ν_L^J be the law of $Y_{\sigma_L^J}$ where $\sigma_L^J = \inf\{u \geq \sigma_L; Y \in \mathcal{S}_L^J$ and $Y_u \leq Y_{\sigma_L}\}$, and let Z_L^J with distribution function F_L^J be the random variable which arises in the Jth level intermediate step in Lemma A.7. Finally let Z_L be the random variable constructed as the limit of Z_L^J.

Suppose M is also a dyadic rational of the form $M = k_K 2^{-\kappa_M}$ with $M < L$. Use a similar set of conventions to define stopping times, distributions and random variables with subscript M. Suppose $J \geq \max\{\kappa_L, \kappa_M\}$. By the remarks after Lemma A.6 (applied $2^J(L-M)$ times) $Z_M^J \leq Z_L^J$. Now letting $J \uparrow \infty$ we find that this property is inherited by the limit variables and hence we may assume $Z_M \leq Z_L$.

Proposition A.8. *Suppose σ is an embedding of ν. There exists a $\mathcal{F}_0$-measurable random variable Z taking values in $[-\infty, \infty]$ such that $\sigma \wedge H_Z^Y$ is minimal, $Y_{\sigma \wedge H_Z^Y}$ has law ν, and $\mathbb{E}[\sigma \wedge H_Z^Y] = E_Y(\infty, \nu)$.*

Proof. By Lemma A.7, for any dyadic L there exists a random variable Z_L such that $\sigma \wedge H_{Z_L}^Y$ is minimal, $Y_{\sigma_L \wedge H_{Z_L}^Y}$ has law ν_L, and $\mathbb{E}[\sigma_L \wedge H_{Z_L}^Y] = \int q_\infty(z)\nu_L(dz)$.

Now consider a sequence of dyadic rationals $(L(j))_{j \geq 1}$ with $L(j) \downarrow \ell$. By the arguments before the proposition $Z_{L(j)}$ is monotonic, $Z_{L(j)} \to Z$ almost surely (where Z is an extended random variable) and $H_{Z_{L(j)}}^Y \to H_Z^Y$ almost surely. Since $\sigma_{L(j)} \downarrow \sigma$ we have $Y_{\sigma_{L(j)} \wedge H_{Z_{L(j)}}^Y} \to Y_{\sigma \wedge H_Z^Y}$ almost surely, and hence $\mathcal{L}(Y_{\sigma \wedge H_Z^Y}) = \lim \nu_{L(j)} = \nu$.

We have $\mathbb{E}[\sigma \wedge H_Z] = \lim \mathbb{E}[\sigma \wedge H_{Z_{L(j)}}]$. Then since $\mathbb{E}[\sigma \wedge H_{Z_{L(j)}}] \leq \mathbb{E}[\sigma_L \wedge H_{Z_{L(j)}}] = E_Y(\infty, \nu_L) \uparrow E_Y(\infty, \nu)$ by the monotnicity of q_∞ we have $\mathbb{E}[\sigma \wedge H_Z] \leq E_Y(\infty, \nu)$. Conversely, by Lemma 4.4, for any embedding ρ of ν we have $\mathbb{E}[\rho] \geq E_Y(\infty, \nu)$. Hence $\mathbb{E}[\sigma \wedge H_Z] = E_Y(\infty, \nu)$ and $\sigma \wedge H_Z$ is minimal as required. □

Corollary A.9. *Suppose σ is an embedding of ν. Suppose $E_Y(\infty, \nu) < \infty$. If σ is minimal then $\mathbb{E}[\sigma] = E_Y(\infty, \nu)$.*

Proof. By Proposition A.8, there exists Z such that $\mathbb{E}[\sigma \wedge H_Z^Y] = E_Y(\infty, \nu)$. But, if σ is minimal then $\sigma = \sigma \wedge H_Z^Y$ and the result follows. □